The Traveling Salesman Problem
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8

Polyhedral theory

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1	INTRODUCTION TO POLYHEDRAL THEORY	251
2	POLYTOPES ASSOCIATED WITH THE SYMMETRIC AND ASYMMETRIC TSP	256
	2.1 The general approach	256
	2.2 The TSP case	257
	2.3 Basic properties of $\tilde{Q}_T^n$ and $\tilde{P}_T^n$	258
	2.4 Elementary properties of tours and related inequalities	259
3	WELL-SOLVABLE COMBINATORIAL OPTIMIZATION PROBLEMS RELATED TO THE TSP	261
	3.1 The symmetric case	262
	3.2 The asymmetric case	265
4	THE SYMMETRIC TRAVELING SALESMAN POLYTOPES	268
	4.1 Some properties of Q_T^n, relations to $\tilde{Q}_T^n$	268
	4.2 Clique tree inequalities	272
	4.3 'Nice' facets of Q_T^n and $\tilde{Q}_T^n$	276
	4.4 'Bad' facets of Q_T^n and $\tilde{Q}_T^n$	281
	4.5 Further remarks	285
5	THE ASYMMETRIC TRAVELING SALESMAN POLYTOPES	290
	5.1 Basic properties of P_T^n and $\tilde{P}_T^n$, sequential lifting	291
	5.2 Relations to the symmetric TSP, new valid inequalities	294
	5.3 Facets of P_T^n and $\tilde{P}_T^n$	297
	5.4 Neighbor relations	302

1 INTRODUCTION TO POLYHEDRAL THEORY

This chapter focuses on polyhedral aspects of the TSP from a theoretical point of view. It lays the foundation for Chapter 9, where algorithmic implications of the polyhedral results are discussed. In particular, it turns out that large classes of facet-defining inequalities can be efficiently identified and can be used as the backbone of computationally successful linear programming based algorithms for TSPs.

All algorithmic problems arising in connection with cutting plane genera-

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tion or facet identification are postponed to Chapter 9. We will deal in this chapter solely with *descriptive* results concerning the facial structure of traveling salesman polytopes.

A detailed treatment of the theory of polyhedra is presented in Bachem & Grötschel [1982], Grünbaum [1967], Rockafellar [1970] and Stoer & Witzgall [1970], as well as in some books on linear programming. For completeness, however, we shall summarize here those concepts and results from linear algebra and polyhedral theory which are necessary for our exposition.

If $x_1, \ldots, x_k \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, then the vector $x \in \mathbb{R}^n$ with $x = \lambda_1 x_1 + \ldots + \lambda_k x_k$ is called a *linear combination* of the vectors $x_1, \ldots, x_k$. If the λ_i in addition satisfy $\lambda_1 + \ldots + \lambda_k = 1$, then x is called an *affine combination* of the vectors $x_1, \ldots, x_k$. If $x = \lambda_1 x_1 + \ldots + \lambda_k x_k$ is an affine combination such that $\lambda_i \geq 0$ for $i = 1, \ldots, k$, then x is called a *convex combination* of the vectors $x_1, \ldots, x_k$.

If $\emptyset \neq S \subseteq \mathbb{R}^n$, then the set of all linear (affine, convex) combinations of finitely many vectors in S is called the *linear* (*affine, convex*) *hull* of S and is denoted by $\text{lin}(S)$ ($\text{aff}(S), \text{conv}(S)$); by convention $\text{lin}(\emptyset) = \{0\}$, $\text{aff}(\emptyset) = \text{conv}(\emptyset) = \emptyset$. A set $S \subseteq \mathbb{R}^n$ with $S = \text{lin}(S)$ ($S = \text{aff}(S)$, $S = \text{conv}(S)$) is called a *linear subspace* (*affine subspace, convex set*).

One can show that a set $L \subseteq \mathbb{R}^n$ is a linear (affine) subspace if and only if there is an (m, n)-matrix A (an (m, n)-matrix A and a vector $b \in \mathbb{R}^m$) such that $L = \{x \in \mathbb{R}^n \mid Ax = 0\}$ ($L = \{x \mid Ax = b\}$). Affine subspaces of particular interest are *hyperplanes*, i.e. sets of the form $\{x \in \mathbb{R}^n \mid a^T x = a_0\}$ where $a \in \mathbb{R}^n - \{0\}$ and $a_0 \in \mathbb{R}$. Clearly, every affine subspace different from $\mathbb{R}^n$ is the intersection of hyperplanes.

A nonempty set $S \subseteq \mathbb{R}^n$ is called *linearly* (*affinely*) *independent*, if for every finite set $\{x_1, x_2, \ldots, x_k\} \subseteq S$, the equations $\lambda_1 x_1 + \ldots + \lambda_k x_k = 0$ ($\lambda_1 x_1 + \ldots + \lambda_k x_k = 0$ and $\lambda_1 + \ldots + \lambda_k = 0$) imply $\lambda_i = 0$, $i = 1, \ldots, k$; otherwise S is called *linearly* (*affinely*) *dependent*. Every linearly (affinely) independent set in $\mathbb{R}^n$ contains at most $n(n+1)$ elements. Moreover, for sets S with at least two elements, linear (affine) independence means that no $x \in S$ can be represented as a linear (affine) combination of the vectors in $S - \{x\}$. All sets $\{x\}, x \neq 0$, are affinely and linearly independent, $\{0\}$ is linearly dependent but affinely independent. By convention, the empty set is linearly and affinely independent.

The *rank* (*affine rank*) of a set $S \subseteq \mathbb{R}^n$ is the cardinality of the largest linearly (affinely) independent subset of S, and the *dimension* of S, denoted by $\dim(S)$, is the affine rank of S minus one. A set $S \subseteq \mathbb{R}^n$ is called *full-dimensional* if $\dim(S) = n$; this is equivalent to saying that there is no hyperplane containing S.

It is clear from the definition that the affine rank of a set is equal to the affine rank of its affine hull. Moreover, if $0 \notin \text{aff}(S)$, i.e. if S is contained in a hyperplane $\{x \mid a^T x = a_0\}$ with $a_0 \neq 0$, then $\dim(S)$ is the maximum cardinality of a linearly independent set in S minus one.

The *rank of a matrix* is the rank of the set of its column vectors (which is the same as the rank of the set of the row vectors of the matrix). An (m, n)-matrix is said to have *full rank* if its rank equals $\min\{m, n\}$.

If S is a subset of $\mathbb{R}^n$, then $Ax = b$ is called a *minimal equation system* for S if $\text{aff}(S) = \{x \in \mathbb{R}^n \mid Ax = b\}$ and A has full rank.

A set $H \subseteq \mathbb{R}^n$ is called a *halfspace* if there is a vector $a \in \mathbb{R}^n$ and a scalar $a_0 \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^n \mid a^T x \leqslant a_0\}$. We say that H is the halfspace defined by the inequality $a^T x \leqslant a_0$, and we shall also say that (if $a \neq 0$) the hyperplane $\{x \mid a^T x = a_0\}$ is the hyperplane defined by $a^T x \leqslant a_0$.

An inequality $a^T x \leqslant b$ is called *valid* with respect to $S \subseteq \mathbb{R}^n$ if $S \subseteq \{x \in \mathbb{R}^n \mid a^T x \leqslant b\}$, i.e. if S is contained in the halfspace defined by $a^T x \leqslant b$. A valid inequality $a^T x \leqslant b$ for S is called *supporting* if $S \cap \{x \in \mathbb{R}^n \mid a^T x = b\} \neq \emptyset$. An inequality $a^T x \leqslant b$ valid with respect to S is called a *proper valid* inequality if S is not contained in the hyperplane $\{x \in \mathbb{R}^n \mid a^T x = b\}$. A valid inequality for S which is not proper is sometimes called an *implicit equation* for S.

A *polyhedron* is the intersection of finitely many halfspaces, i.e. every polyhedron P can be represented in the form $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$. Since an equation system $Dx = c$ can be written as $Dx \leqslant c$, $-Dx \leqslant -c$, every set of the form $\{x \in \mathbb{R}^n \mid Ax \leqslant b, Dx = c\}$ is a polyhedron. A bounded polyhedron (i.e. a polyhedron P with $P \subseteq \{x \in \mathbb{R}^n \mid \|x\| \leqslant B\}$ for some $B > 0$ where $\|x\|$ is, for example, the Euclidean norm of x) is called a *polytope*. Polytopes are precisely those sets in $\mathbb{R}^n$ which are the convex hulls of finitely many points, i.e. every polytope P can be written as $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ (A an (m, n)-matrix, $b \in \mathbb{R}^m$), and as $P = \text{conv}(V)$ ($V \subseteq \mathbb{R}^n$, $|V|$ finite).

A subset F of a polyhedron P is called a *face* of P if there exists an inequality $a^T x \leqslant a_0$ valid with respect to P such that $F = \{x \in P \mid a^T x = a_0\}$. We say that the inequality $a^T x \leqslant a_0$ defines F. A face F is called *proper* if $F \neq P$. In fact, if $P = \{x \in \mathbb{R}^n \mid a_i^T x \leqslant b_i, i = 1, \ldots, k\}$ is a polyhedron and F is a face of P, then one can show that there exists an index set $I \subseteq \{1, \ldots, k\}$ such that $F = \{x \in P \mid a_i^T x = b_i, i \in I\}$. Similarly, if $P = \text{conv}(V)$ for a finite set $V \subseteq \mathbb{R}^n$ and if F is a face of the polytope P, then there exists a set $W \subseteq V$ with $F = \text{conv}(W)$.

If $a^T x \leqslant a_0$ and $b^T x \leqslant b_0$ are two valid inequalities for a polyhedron P and if $\{x \in P \mid a^T x = a_0\} = \{x \in P \mid b^T x = b_0\}$ (i.e. both inequalities 'define' or 'induce' the same face), we say that $a^T x \leqslant a_0$ and $b^T x \leqslant b_0$ are *equivalent* with respect to P.

A face which contains one element only is called a *vertex*. If $\{x\}$ is a vertex of P we shall simply say that x is a vertex of P. (The word 'vertex' is standard in polyhedral theory as well as in graph theory, so we will use it in two meanings. We made sure that there will be no confusion.) A *facet* F of a polyhedron P is a proper, nonempty face (i.e. a face satisfying $\emptyset \neq F \neq P$) which is maximal with respect to set inclusion.

Clearly, the set of feasible solutions of a linear programming problem $\max\{c^T x \mid Ax \leqslant b\}$ forms a polyhedron P. If c_0 is the optimum value of

$\max\{c^T x \mid x \in P\}$, then $c^T x \leqslant c_0$ is a supporting valid inequality for P, i.e. the set $F = \{x \in P \mid c^T x = c_0\}$ of optimum solutions is a face of P. If P contains a vertex, then every face contains a vertex. This implies in particular that every linear program over a polytope has at least one optimum vertex solution.

In order to apply linear programming techniques (the simplex method, the ellipsoid method, Karmarkar's method [1985], relaxation methods, etc.) polyhedra have to be given in the form $\{x \in \mathbb{R}^n \mid Ax \leqslant b\}$. In combinatorial optimization, however, polyhedra are usually given as the convex hulls of finite sets of points; thus a major problem is to find an inequality system defining such a polyhedron. Moreover, one wants to find inequality systems with as few inequalities as possible. For these purposes facets, i.e. facet-defining inequalities, are of particular importance.

Theorem 1 *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and assume that A is an (m, n)-matrix, $b \in \mathbb{R}^m$ such that $\operatorname{aff}(P) = \{x \in \mathbb{R}^n \mid Ax = b\}$. Let F be a nonempty face of P, then the following statements are equivalent:*

(a) *F is a facet of P.*

(b) *F is a maximal proper face of P.*

(c) $\dim(F) = \dim(P) - 1$.

(d) *There exists an inequality $c^T x \leqslant c_0$ valid with respect to P with the following three properties:*

  (d_1) $F \subseteq \{x \in P \mid c^T x = c_0\}$.

  (d_2) *There exists $\bar{x} \in P$ with $c^T \bar{x} < c_0$, i.e. the inequality is proper.*

  (d_3) *If any other inequality $d^T x \leqslant d_0$ valid with respect to P satisfies $F \subseteq \{x \in P \mid d^T x = d_0\}$, then there exists a scalar $\alpha \geqslant 0$ and a vector $\lambda \in \mathbb{R}^m$ such that*

$$d^T = \alpha c^T + \lambda^T A,$$
$$d_0 = \alpha c_0 + \lambda^T b.$$

Conditions (c) and (d) provide the two basic methods to prove that a given inequality $c^T x \leqslant c_0$ defines a facet of a polyhedron P. In both cases, of course, one first has to check that $c^T x \leqslant c_0$ is valid and that P is not contained in $\{x \in \mathbb{R}^n \mid c^T x = c_0\}$. This is usually trivial.

The first method consists of exhibiting a set of $k = \dim(P)$ vectors (usually vertices of P) $x_1, x_2, \ldots, x_k \in P$ satisfying $c^T x_i = c_0$, $i = 1, \ldots, k$, and showing that these vectors are affinely independent. (If $c_0 \neq 0$, this is equivalent to showing that these k vectors are linearly independent.) Let us call this method the *direct method*. We shall encounter some cases where the direct method is easy to apply since the linear (or affine) independence of 'simply structured' vectors like unit vectors and simple modifications of these is easy to check.

In most (nontrivial) cases the second *indirect method*, based on condition (d) of Theorem 1, is more suitable, and it is as follows. One assumes the existence of a valid inequality $d^T x \leqslant d_0$ with $\{x \in P \mid c^T x = c_0\} \subseteq \{x \in P \mid d^T x = d_0\}$. Using the known equation system $Ax = b$ for P, one can determine a $\lambda \in \mathbb{R}^m$ such that $\bar{d} := d + A^T\lambda$ has certain useful properties, i.e. some of the coefficients of $\bar{d}$ are equal to the corresponding coefficients of the given c or the like. Then utilizing known properties of the points x in P satisfying $c^Tx = c_0$, one determines the still unknown coefficients of $\bar{d}$ iteratively. If it turns out that $\bar{d} = \alpha c + A^T\mu$ for some $\alpha \geq 0$ and $\mu \in \mathbb{R}^m$, then condition (d) of Theorem 1 implies that $c^Tx \leq c_0$ defines a facet of P.

Facets are of importance since they have to be known in order to obtain a minimal inequality representation of a polyhedron. Let $P \neq \mathbb{R}^n$ be a polyhedron; then a system of equations and inequalities $Dx = c$, $Ax \leq b$ is said to be *complete* with respect to P if $P = \{x \in \mathbb{R}^n \mid Dx = c, Ax \leq b\}$. (The equation system may be vacuous.) Let us call such a system *nonredundant* if $Ax \leq b$ contains no implicit equations and if the deletion of any equation or inequality of the system results in a polyhedron different from P. Any equation or inequality which can be deleted without changing the polyhedron is called *redundant.*

Theorem 2 *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $Ax \leq b$, $Dx = c$ be a complete and nonredundant system for P, where D is an (m, n)-matrix and A is a (k, n)-matrix. Then the following hold:*

(a) $\text{aff}(P) = \{x \in \mathbb{R}^n \mid Dx = c\}$ *and* $m = \text{rank}(D)$.

(b) $\text{aff}(P)$ *and P have dimension $n - m$.*

(c) *Every inequality $a_i^T \leq b_i$ of the system $Ax \leq b$ defines a facet F_i of P, where $F_i = \{x \in P \mid a_i^Tx = b_i\}$, $i = 1, \ldots, k$.*

(d) *If* $\bar{a}_i^Tx \leq \bar{b}_i$, $i = 1, \ldots, \bar{k}$,
$\bar{d}_i^Tx = \bar{c}_i$, $i = 1, \ldots, \bar{m}$,
is any other complete and nonredundant system for P, then
 - (d_1) $k = \bar{k}$, $m = \bar{m}$,
 - (d_2) $\bar{d}_i^T = (\lambda^i)^TD$ *for some* $\lambda^i \in \mathbb{R}^m - \{0\}$ $(i = 1, \ldots, m)$,
 - (d_3) $\bar{a}_i^T = \alpha_i a_j^T + (\lambda^i)^TD$ *for some* $\alpha_i > 0$, $\lambda^i \in \mathbb{R}^m$, *and* $j \in \{1, \ldots, k\}$ $(i = 1, \ldots, \bar{k})$.

Theorem 2(d) in particular implies that for a full-dimensional polyhedron P there is a complete and nonredundant inequality system $a_i^Tx \leq b_i$, $i = 1, \ldots, k$, such that every complete and nonredundant inequality system $\bar{a}_i^Tx \leq \bar{b}_i$, $i = 1, \ldots, \bar{k}$, satisfies $k = \bar{k}$ and $\bar{a}_i = \alpha_i a_i$ for some $\alpha_i > 0$ (after suitable indexing) and $i = 1, \ldots, k$. This justifies the statement that a full-dimensional polyhedron is defined by a *unique* (up to multiplication by positive scalars) nonredundant and complete inequality system. Moreover, for every facet F of P there is a unique (up to multiplication by a positive scalar) inequality defining F.

Suppose a polytope P is given as the convex hull of finitely many points (we will encounter such polytopes in the following), then Theorem 2(c) implies that in order to get a complete inequality description of P, for every facet of P one has to know (at least) one inequality defining it. Moreover, if we want to find a complete and nonredundant system $Ax \leq b$, $Dx = c$ for P, we have to prove that $Dx = c$ is a minimal equation system for P, that

$Ax \leqslant b$ contains no implicit equations, that every inequality of $Ax \leqslant b$ defines a facet of P and that $Ax \leqslant b$ contains no equivalent inequalities, i.e. different inequalities defining the same facet of P.

The main purpose of this chapter is to introduce several large classes of inequalities which are valid for the polytope(s) associated with the TSP and to prove that these inequalities define nonequivalent facets of the polytopes. This will show that incredibly large numbers of inequalities are necessary to give a complete (and nonredundant) description of the traveling salesman polytopes.

2 POLYTOPES ASSOCIATED WITH THE SYMMETRIC AND ASYMMETRIC TSP

2.1 The general approach

The approach that we are going to describe here consists of associating polytopes with the TSP and other closely related problems. This approach is applicable to almost all other combinatorial optimization problems as well; see e.g. Padberg [1979] for a related survey concerning the facial structure of polyhedra related to covering, packing and knapsack problems. The area of research in which polyhedra related to combinatorial optimization problems are investigated is often referred to as 'polyhedral combinatorics' and its principal ideas are discussed next. (For general surveys of this subject see, e.g., Grötschel [1984], Pulleyblank [1983] and Schrijver [1983].)

Let E be a finite ground set and $\mathscr{I}$ be a set of subsets of E. With every element $e \in E$ we associate a variable x_e, i.e. a component of a vector $x \in \mathbb{R}^E$ indexed by e. (Rather than writing $\mathbb{R}^{|E|}$ we simply write $\mathbb{R}^E$.) With every subset $F \subseteq E$ we associate a vector $x^F \in \mathbb{R}^E$, called the *incidence* or *characteristic vector* of F, defined as follows:

$$x_e^F = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{if } e \notin F. \end{cases}$$

Thus, every subset $F \subseteq E$ corresponds to a unique 0–1 vector in $\mathbb{R}^E$ and vice versa. Now we associate with $\mathscr{I} \subseteq 2^E$ the polytope $P_{\mathscr{I}}$ which is the convex hull of all incidence vectors of elements of $\mathscr{I}$, i.e.

$$P_{\mathscr{I}} := \text{conv}\{x^F \in \mathbb{R}^E \mid F \in \mathscr{I}\}. \tag{1}$$

It is easy to see that every vertex of $P_{\mathscr{I}}$ corresponds to a set in $\mathscr{I}$ and vice versa.

Now suppose 'weights' or 'distances' $c_e \in \mathbb{R}$ for all $e \in E$ are given and we want to find $F^* \in \mathscr{I}$ such that $c(F^*) := \sum_{e \in F^*} c_e$ is as small (or as large) as possible. Then we can solve this combinatorial optimization problem via the linear programming problem

$$\min\{c^T x \mid x \in P_{\mathscr{I}}\}, \tag{2}$$

since every optimum solution of the combinatorial optimization problem corresponds to an optimum vertex solution of (2) and vice versa. In order to apply linear programming techniques we need a complete (and, preferably, nonredundant) description of the polytope $P_{\mathscr{I}}$ by way of linear equations and inequalities. As we shall indicate later, such a *completeness* result will probably prove to be elusive for all $\mathcal{NP}$-complete problems, i.e. there is little hope that complete and nonredundant systems of linear equations and inequalities describing $P_{\mathscr{I}}$ will ever be found explicitly for 'hard' combinatorial optimization problems. But we shall also see that *partial* results can be of great computational help for the numerical solution of hard problems when used in conjunction with linear programming and branch and bound methods.

2.2 The TSP case

With respect to the symmetric TSP, the 'natural' polytopes to work with are the following. Let $K_n = (V, E)$ denote the complete graph on n vertices, i.e. for every two different vertices i and j there is exactly one edge $\{i, j\}$ linking i and j. Denote by $\mathscr{S}_n$ the set of all *tours* (*Hamiltonian cycles*) in K_n (note that we view cycles as sets of edges) and let $\tilde{\mathscr{S}}_n$ be the set of all subsets of tours, i.e. $\tilde{\mathscr{S}}_n = \{S \subseteq E \mid \text{there exists a tour } T \subseteq E \text{ with } S \subseteq T\}$. Then the polytope

$$Q_{\mathrm{T}}^n := \operatorname{conv}\{x^T \in \mathbb{R}^E \mid T \in \mathscr{S}_n\}$$

is called the (n-city) *symmetric traveling salesman polytope* and the polytope

$$\tilde{Q}_{\mathrm{T}}^n := \operatorname{conv}\{x^S \in \mathbb{R}^E \mid S \in \tilde{\mathscr{S}}_n\}$$

is called the (n-city) *monotone symmetric traveling salesman polytope.*

In the asymmetric case, we get two such canonical polytopes in the following way. Let $D_n = (V, A)$ be the complete digraph on n vertices, i.e. every two different vertices i and j are linked by two antiparallel arcs (i, j) and (j, i), let $\mathscr{T}_n$ denote the set of all (directed) *tours* (directed Hamiltonian cycles) in D_n, and let $\tilde{\mathscr{T}}_n$ denote the set of all subsets of tours in $\mathscr{T}_n$. Then

$$P_{\mathrm{T}}^n := \operatorname{conv}\{x^T \in \mathbb{R}^A \mid T \in \mathscr{T}_n\}$$

is called the (n-city) *asymmetric traveling salesman polytope* and

$$\tilde{P}_{\mathrm{T}}^n := \operatorname{conv}\{x^S \in \mathbb{R}^A \mid S \in \tilde{\mathscr{T}}_n\}$$

is called the (n-city) *monotone asymmetric traveling salesman polytope.*

Every symmetric TSP can be solved – in principle – via the linear programming problem

$$\min\{c^T x \mid x \in Q_{\mathrm{T}}^n\}, \tag{3}$$

and every asymmetric TSP via the linear programming problem

$$\min\{c^T x \mid x \in P_{\mathrm{T}}^n\}. \tag{4}$$

The monotone polytopes $\tilde{Q}^n_T$ and $\tilde{P}^n_T$ can be used for these purposes as well, in the sense that, if the distance c_{ij} of every edge $\{i, j\}$ or arc (i, j) is replaced by the distance $\tilde{c}_{ij} := M - c_{ij}$, where $M = \max\{|c_{ij}| : \{i, j\} \in E \ \ ((i, j) \in A$, respectively$)\} + 1$, then the resulting maximization problems over $\tilde{Q}^n_T$ and $\tilde{P}^n_T$ provide the same answers as (3) and (4), respectively.

2.3 Basic properties of $\tilde{Q}^n_T$ and $\tilde{P}^n_T$

If E is a finite set and $\mathcal{I}$ a nonempty system of subsets of E, then $\mathcal{I}$ is called *monotone* (or *subclusive*, or *lower comprehensive*, or *hereditary system*, or *independence system*) if $J \in \mathcal{I}$ and $I \subseteq J$ imply $I \in \mathcal{I}$. If $\mathcal{I} \subseteq 2^E$, then $\tilde{\mathcal{I}} = \{I \subseteq E \mid \exists J \in \mathcal{I}$ with $I \subseteq J\}$ is called the *monotonization* of $\mathcal{I}$. Monotonization often preserves important properties of the original system and, at the same time, makes a problem easier to analyze.

A polyhedron $P \subseteq \mathbb{R}^n_+$ (i.e. P is contained in the nonnegative orthant) is called *monotone* if $y \in P$ and $0 \leq x \leq y$ imply $x \in P$. Clearly, if $P_{\mathcal{I}} \subseteq \mathbb{R}^E$ is a polytope associated with a set of subsets on a finite set E, then the polytope $P_{\tilde{\mathcal{I}}} \subseteq \mathbb{R}^E$ associated with the monotonization $\tilde{\mathcal{I}}$ of $\mathcal{I}$ is a monotone polytope. This implies that the traveling salesman polytopes $\tilde{Q}^n_T$ and $\tilde{P}^n_T$ are monotone.

By going from a polytope to its monotonization, we enlarge the polytope 'below'. The advantage of this is that, if $E = \bigcup \mathcal{I}$, we get a full-dimensional polytope which is technically easier to handle. Moreover, a full-dimensional polytope has a unique complete and nonredundant inequality system describing it. This implies in particular that the problem of equivalence of inequalities is easy to solve. Thus, if the monotone polytope is still sufficiently closely related to the original problem, it is often preferable to study the monotone polytope. For the TSP, sufficient closeness is assured by Exercise 1. In the following we shall study the natural TSP-polytopes Q^n_T, P^n_T as well as their monotone versions, $\tilde{Q}^n_T$, $\tilde{P}^n_T$.

Proposition 1 *Let E be a finite set, let $\mathcal{I}$ be an independence system on E, and let $F = E - \bigcup \mathcal{I}$. Then the dimension of $P_{\mathcal{I}}$ is $|E| - |F|$.*

Proof $P_{\mathcal{I}}$ is contained in $\{x \in \mathbb{R}^E \mid x_f = 0, f \in F\}$ and thus $\dim(P_{\mathcal{I}}) \leq |E| - |F|$ holds. On the other hand, $P_{\mathcal{I}}$ contains the zero vector and the unit vector u_e for all $e \in E - F$. Hence, $P_{\mathcal{I}}$ contains $|E| - |F| + 1$ affinely independent vectors, and we are done. □

Corollary 1

$$\dim(\tilde{Q}^n_T) = |E| = n(n-1)/2, \qquad \text{for } n \geq 3.$$
$$\dim(\tilde{P}^n_T) = |A| = n(n-1), \qquad \text{for } n \geq 2.$$

Every polytope $P_{\mathcal{I}}$, cf. (1), is contained in the unit hypercube; thus the hypercube constraints (from now on called *trivial inequalities*) $0 \leq x_e \leq 1$, $e \in E$, are valid inequalities for $P_{\mathcal{I}}$. In fact – except for the obvious case – the nonnegativity constraints always define facets of monotone polytopes.

Lemma 1 *Let $\mathcal{I}$ be an independence system on a finite set E and $P_{\mathcal{I}}$ be the corresponding polytope, then $x_e \geq 0$ defines a facet of $P_{\mathcal{I}}$ if and only if $e \in \bigcup \mathcal{I}$.*

Again, since $E = \bigcup \mathcal{I}$ holds in the TSP case, we obtain the following.

Corollary 2

(a) *For all edges $\{i, j\}$ in K_n, $x_{ij} \geq 0$ defines a facet of $\tilde{Q}^n_T$, $n \geq 3$.*

(b) *For all arcs (i, j) in D_n, $x_{ij} \geq 0$ defines a facet of $\tilde{P}^n_T$, $n \geq 2$.* □

Note that, above, a variable corresponding to an edge $\{i, j\}$ or an arc (i, j) should actually be written as $x_{\{i,j\}}$ or $x_{(i,j)}$. For notational convenience we drop the brackets and, in most cases, also the comma. We shall keep this notation in the following as well as in the next chapter. Thus in the symmetric case, the variables x_{ij} and x_{ji} are identical (but not in the asymmetric case).

The trivial inequalities $x_e \leq 1$ do not necessarily define facets, not even for monotone polytopes, i.e. here every problem has to be checked individually.

Proposition 2 *For every edge $\{i, j\}$ in K_n the inequality $x_{ij} \leq 1$ defines a facet of $\tilde{Q}^n_T$, $n \geq 3$.*

Proof For this proof we use the direct method suggested by Theorem 1(c). The $|E|$ edge sets $\{\{i, j\}\}$ and $\{\{i, j\}, \{p, q\}\}$, where $\{p, q\} \in E - \{i, j\}$, are obviously contained in $\tilde{\mathcal{S}}_n$. Their incidence vectors satisfy $x_{ij} \leq 1$ with equality and are linearly independent. Moreover, there is a set in $\tilde{\mathcal{S}}_n$ whose incidence vector does not satisfy $x_{ij} \leq 1$ with equality. Thus by Theorem 1, $x_{ij} \leq 1$ defines a facet of $\tilde{Q}^n_T$. □

In fact, it is easy to see that $\tilde{Q}^3_T$ equals the unit hypercube in $\mathbb{R}^3$ and $\tilde{P}^2_T$ equals the unit hypercube in $\mathbb{R}^2$ (see also Exercise 3). These are the only cases where the trivial inequalities are sufficient to describe a traveling salesman polytope.

2.4 Elementary properties of tours and related inequalities

The feasible solutions of combinatorial optimization problems are usually 'structured' in some way. From such structural properties, inequalities which are valid for the associated polytope can often be derived in a straightforward manner.

In the symmetric case, a tour has the property that every vertex lies on exactly two edges, and hence if S is a subset of a tour, then every vertex lies on at most two edges in S. Let $\delta(v)$ be the *star* (or cut) of v, i.e. the set of edges in K_n having v as one endpoint, then our foregoing observation implies that

$$x(\delta(v)) = 2 \qquad \text{for all } v \in V \tag{5}$$

is a system of n equations satisfied by all incidence vectors of tours. (Note that above and in the following $x(F)$ is used as an abbreviation of $\sum_{e \in F} x_e$ where F is an arc or edge set.) This implies that $Q^n_T \subseteq \{x \in \mathbb{R}^E \mid x$ satisfies

(5)}. Similarly, we obtain that every incidence vector of a subset of a tour satisfies the n inequalities

$$x(\delta(v)) \leqslant 2 \qquad \text{for all } v \in V. \tag{6}$$

Thus we have that $\tilde{Q}^n_T \subseteq \{x \in \mathbb{R}^E \mid x \text{ satisfies } (6)\}$.

In the asymmetric case we observe that every tour has the property that every vertex in D_n is the head of exactly one arc and the tail of exactly one arc. Let $\overleftarrow{\delta}(v)$, $\vec{\delta}(v)$, respectively, denote the set of arcs having head v, and having tail v, respectively. Then the system

$$\begin{aligned} x(\overleftarrow{\delta}(v)) &= 1 \qquad \text{for all } v \in V, \\ x(\vec{\delta}(v)) &= 1 \qquad \text{for all } v \in V, \end{aligned} \tag{7}$$

of $2n$ equations is satisfied by every vector in P^n_T, whereas every point in $\tilde{P}^n_T$ satisfies

$$\begin{aligned} x(\overleftarrow{\delta}(v)) &\leqslant 1 \qquad \text{for all } v \in V, \\ x(\vec{\delta}(v)) &\leqslant 1 \qquad \text{for all } v \in V. \end{aligned} \tag{8}$$

This implies $P^n_T \subseteq \{x \in \mathbb{R}^A \mid x \text{ satisfies } (7)\}$ and $\tilde{P}^n_T \subseteq \{x \in \mathbb{R}^A \mid x \text{ satisfies } (8)\}$. The equations (5), (7) and the inequalities (6), (8) are called *degree constraints* since they restrict the possible degree (indegree, outdegree) of a vertex of the graph (digraph) in a feasible solution.

Note that the fact that a tour contains n edges (n arcs, respectively) gives no additional polyhedral information since this is implied by (5) (by (7), respectively).

A further obvious property of a tour is that it is a (directed) cycle. This means that a tour or a subset of a tour contains no cycle of length less than n. From this observation we can derive, for example in the symmetric case, that

$$x(C) \leqslant |C| - 1 \tag{9}$$

is a valid inequality for Q^n_T and $\tilde{Q}^n_T$, where C is the edge set of a cycle in K_n of length less than n. Such a cycle inequality (9) can be improved by the following observation. If $W \subseteq V$ is a set of vertices with $2 \leqslant |W| \leqslant n-1$, then the set of edges $E(W)$ (i.e. the set of edges in K_n with both endpoints in W) intersects every tour in at most $|W|-1$ edges, because every set of more than $|W|-1$ edges contained in the induced subgraph $(W, E(W))$ contains at least one cycle. Consequently, the system of inequalities

$$x(E(W)) \leqslant |W| - 1 \qquad \text{for all } W \subseteq V, \qquad 2 \leqslant |W| \leqslant n-1 \tag{10}$$

is satisfied by all points in Q^n_T and $\tilde{Q}^n_T$. The inequalities (10) are the well-known *subtour elimination constraints* introduced by Dantzig, Fulkerson & Johnson [1954]. Note that for $|W|=2$, the inequality $x(E(W)) \leqslant 1$ is nothing but a trivial inequality $x_{ij} \leqslant 1$.

In the asymmetric case, for $W \subseteq V$ let $A(W)$ denote the set of arcs in D_n

having head and tail in W. The same observation as above implies that the *subtour elimination constraints*

$$x(A(W)) \leqslant |W|-1 \qquad \text{for all } W \subseteq V, \qquad 2 \leqslant |W| \leqslant n-1 \tag{11}$$

are valid inequalities for P_T^n and $\tilde{P}_T^n$.

We shall study the equations and inequalities introduced above in more detail in a later section.

The equations and inequalities (5), (6), (7) and (8) can be used to show the following.

Lemma 2
(a) *Q_T^n is a face of $\tilde{Q}_T^n$.*
(b) *P_T^n is a face of $\tilde{P}_T^n$.*

This implies the following observation. (Why?)

Corollary 3 *Every inequality valid for $\tilde{Q}_T^n$ (resp. $\tilde{P}_T^n$) is also valid for Q_T^n (resp. P_T^n).*

Note that Corollary 3 implies that every complete linear description of $\tilde{Q}_T^n$ or $\tilde{P}_T^n$ yields a complete description of Q_T^n or P_T^n.

Exercises

1. Given a n-city symmetric (asymmetric) TSP with distance function c, prove that every optimum vertex solution of $\max\{\bar{c}^T x \mid x \in \tilde{Q}_T^n\}$ (of $\max\{\bar{c}^T x \mid x \in \tilde{P}_T^n\}$, respectively) corresponds to an optimum tour and vice versa (for the definition of $\bar{c}$ see the end of Section 2.2).
2. Prove Lemma 1. (*Hint*: see Hammer, Johnson & Peled [1975].)
3. Prove that for $n \geqslant 3$ no inequality $x_{ij} \leqslant 1$ defines a facet of $\tilde{P}_T^n$. (*Hint*: consider subtour elimination constraints (11) when $|W| = 2$.)
4. Prove Lemma 2.
5. Does every inequality which defines a facet of $\tilde{Q}_T^n$ (of $\tilde{P}_T^n$) also define a facet of Q_T^n (of P_T^n)? What about the other way around?

3 WELL-SOLVABLE COMBINATORIAL OPTIMIZATION PROBLEMS RELATED TO THE TSP

Solution methods for $\mathcal{NP}$-complete combinatorial optimization problems, such as branch and bound, are usually based on relaxations of the problem which are easy to solve. A relaxation in this case is another problem (often itself a combinatorial optimization problem) which has the property that every feasible solution of the original problem corresponds (in a unique way) to a feasible solution of the relaxed problem. As we shall see below, relaxations can also be used to obtain polyhedral information about the problem in question. We do not discuss in this section the several known special cases of the TSP which are well solved. For a discussion of these the reader is referred to Chapter 4.

3.1 The symmetric case

A *forest* in a graph $G=(V,E)$ is a set of edges containing no cycle; a *spanning tree* is a forest with $|V|-1$ edges. It is well known that by adding an edge to a spanning tree, exactly one cycle is created. Thus, a spanning tree plus an edge is an edge set containing n edges and exactly one cycle. Hence, every tour is a spanning tree plus an edge.

Let us consider a slightly more special construction. Consider the complete graph $K_n=(V,E)$ and assume that the vertices are labeled $1,2,\dots,n$. Call an edge set S a *1-tree* (in K_n) if $|S\cap\delta(1)|=2$ and $S\cap E(\{2,\dots,n\})$ is a spanning tree in $K_n-\{1\}$. That is, a 1-tree is a set of n edges containing exactly one cycle which contains vertex 1. Thus every tour is a 1-tree. By calculating a minimum spanning tree on the vertices $2,\dots,n$ and choosing the two shortest edges containing vertex 1, it is easy to find a minimum weight 1-tree in K_n. Hence the minimum 1-tree problem is an (often used and reasonable) relaxation of the symmetric TSP.

The *1-tree polytope* Q^n_{1T} is the convex hull of all incidence vectors of 1-trees in K_n, i.e.

$$Q^n_{1T}=\operatorname{conv}\{x^S\subseteq\mathbb{R}^E \mid S \text{ is a 1-tree in } K_n\},$$

and clearly we have $Q^n_T\subseteq Q^n_{1T}$. Let $\tilde{\mathcal{I}}$ denote the set of all subsets of 1-trees in K_n and

$$\tilde{Q}^n_{1T}=\operatorname{conv}\{x^S\in\mathbb{R}^E \mid S\in\tilde{\mathcal{I}}\},$$

then one can easily see that $(E,\tilde{\mathcal{I}})$ is a *matroid* (i.e. an independence system satisfying: $I,J\in\tilde{\mathcal{I}}$, $|I|<|J|\Rightarrow\exists e\in J-I$ such that $I\cup\{e\}\in\tilde{\mathcal{I}}$).

Edmonds [1971] has shown how a complete and nonredundant system describing a polytope associated with a matroid can be constructed; see also Giles [1975]. In the case of 1-trees in K_n, $n\geqslant 3$, such a system describing $\tilde{Q}^n_{1T}$ is the following:

$$0\leqslant x_e\leqslant 1 \qquad \text{for all } e\in E, \tag{12}$$

$$x(\delta(1))\leqslant 2, \tag{13}$$

$$x(E(W))\leqslant |W|-1 \qquad \text{for all } W\subseteq V,\ |W|\geqslant 3,\ 1\notin W. \tag{14}$$

A complete and nonredundant system for Q^n_{1T} can be derived easily now (see Held & Karp [1970], and for general techniques similar to this one see Giles [1975] and Grötschel [1977a]):

$$0\leqslant x_e\leqslant 1 \qquad \text{for all } e\in E, \tag{15}$$

$$x(\delta(1))=2, \tag{16}$$

$$x(E(\{2,\dots,n\}))=n-2, \tag{17}$$

$$x(E(W))\leqslant|W|-1 \qquad \text{for all } W\subseteq V,\ 3\leqslant|W|\leqslant|V|-2,\ 1\notin W. \tag{18}$$

In other words, the vertices of the polyhedron given by (15), (16), (17), (18) are the incidence vectors of 1-trees in K_n. Note that the inequalities (14), (18) respectively, are nothing but subtour elimination constraints (10). Moreover, we know that the inequalities (12), (13), (14), ((15), (18), respectively) define facets of $\tilde{Q}^n_{1T}$ (of Q^n_{1T}, respectively) and that no two of these inequalities are equivalent.

A second interesting relaxation of the symmetric TSP is the 2-matching problem. A *2-matching* (*perfect 2-matching*) in a graph is a set of edges such that every vertex is contained in at most (exactly) two edges. Clearly, every tour (subset of a tour) is a perfect 2-matching (a 2-matching). Denote by Q^n_{2M} (by $\tilde{Q}^n_{2M}$, respectively) the *perfect 2-matching polytope* (the *2-matching polytope*, respectively) of K_n, where $n \geq 3$, i.e.

$$Q^n_{2M} = \operatorname{conv}\{x^M \in \mathbb{R}^E \mid M \text{ is a perfect 2-matching in } K_n\},$$
$$\tilde{Q}^n_{2M} = \operatorname{conv}\{x^M \in \mathbb{R}^E \mid M \text{ is a 2-matching in } K_n\}.$$

$\tilde{Q}^n_{2M}$ is the monotonization of Q^n_{2M} and $\tilde{Q}^n_T \subseteq \tilde{Q}^n_{2M}$ and $Q^n_T \subseteq Q^n_{2M}$ hold.

Edmonds [1965c] has given a result which includes a complete linear description of $\tilde{Q}^n_{2M}$ and Q^n_{2M}. From these descriptions, nonredundant characterizations of $\tilde{Q}^n_{2M}$ and Q^n_{2M} (for K_n only) were derived by Grötschel [1977b] and Grötschel [1977a], respectively. In order to show how the *2-matching inequalities* of Edmonds can be generalized further, we introduce them in a form different from the usual one.

Let $K_n = (V, E)$ be the complete graph on $n \geq 3$ vertices and assume that $H, T_1, T_2, \ldots, T_k \subseteq V$, $k \geq 1$, are vertex sets satisfying

$$|H \cap T_i| = 1, \qquad i = 1, \ldots, k,$$
$$|T_i - H| = 1, \qquad i = 1, \ldots, k,$$

then the *2-matching inequality*

$$x(E(H)) + \sum_{i=1}^{k} x(E(T_i)) \leq |H| + \left\lfloor \frac{k}{2} \right\rfloor \tag{19}$$

is valid for $\tilde{Q}^n_{2M}$ and Q^n_{2M}. It is easy to see that if there is no round-down in (19), i.e. if k is even, a 2-matching inequality is redundant with respect to Q^n_{2M} and $\tilde{Q}^n_{2M}$. Edmonds [1965c] has proved that the inequalities (19) with odd k yield a complete linear description. In fact, we have the following slightly stronger result.

Theorem 3 *The following system of inequalities is a complete and non-redundant characterization of the 2-matching polytope $\tilde{Q}^n_{2M}$, $n \geq 4$:*

$$0 \leq x_e \leq 1 \qquad \textit{for all } e \in E, \tag{20}$$

$$x(\delta(x)) \leq 2 \qquad \textit{for all } v \in V, \tag{21}$$

$$x(E(H)) + \sum_{i=1}^{k} x(E(T_i)) \leq |H| + \frac{k-1}{2} \qquad \textit{for all } H, T_1, \ldots, T_k \subseteq V, \tag{22}$$

satisfying
(a) $|H \cap T_i| = 1, \ i = 1, \ldots, k,$
(b) $|T_i - H| = 1, \ i = 1, \ldots, k,$
(c) $T_i \cap T_j = 0, \quad 1 \leqslant i < j \leqslant k,$
(d) $k \geqslant 3$ *and odd, or* $k = 1$ *and* $|H| \geqslant 4$.

The polytope Q^3_{2M} is the unit hypercube in $\mathbb{R}^3$.

In the case of perfect 2-matchings the inequalities (21) must be stated as equations. Then one can show the following result.

Lemma 3 *Let* $H, T_1, \ldots, T_k \subseteq V$ *and* $H', T'_1, \ldots, T'_{k'}$ *be two different systems of vertex sets satisfying* (22)(a), ..., (d). *Then the 2-matching inequalities* (22) *corresponding to* $H, T_1, \ldots, T_k$ *and* $H', T'_1, \ldots, T'_{k'}$ *are equivalent with respect to* Q^n_{2M} *if and only if*
(a) $k = k'$,
(b) *for every* $i \in \{1, \ldots, k\}$ *there is a* $j \in \{1, \ldots, k'\}$ *with* $T_i = T'_j$,
(c) $H' = V - H$.

The degree equations $x(\delta(v)) = 2$, $v \in V$, form a minimal equation system for Q^n_{2M} and the rank of the matrix corresponding to these equations is n. In fact, one can prove that the degree equations determine the affine hull of Q^n_{2M} [Grötschel 1977a]. Combining this observation with Theorem 3 and Lemma 3, one can show the following result.

Theorem 4 *Let* Q^n_{2M} *be the perfect 2-matching polytope for* $K_n = (V, E)$, $n \geqslant 5$. *Then*

$$\dim Q^n_{2M} = |E| - n.$$

Let $\mathcal{V}$ *be any set of subsets of* V *with* $W \in \mathcal{V}$ *if and only if* $V - W \notin \mathcal{V}$. *Then the following system of equations and inequalities is a complete and nonredundant description of* Q^n_{2M}:

$$0 \leqslant x_e \leqslant 1 \qquad \textit{for all } e \in E, \tag{23}$$

$$x(\delta(v)) = 2 \qquad \textit{for all } v \in V, \tag{24}$$

$$x(E(H)) + \sum_{i=1}^{k} x(E(T_i)) \leqslant |H| + \frac{k-1}{2} \tag{25}$$

for all $H, T_1, \ldots, T_k \subseteq V$ *satisfying* (a), (b), (c) *of* (22) *and*
(d′) ($k \geqslant 3$ *and* k *odd*) *or* ($k = 1$ *and* $4 \leqslant |H| \leqslant n - 4$),
(e) $H \in \mathcal{V}$.

A consequence of the results of this section is that

$$\tilde{Q}^n_T \subseteq \tilde{Q}^n_{1T} \cap \tilde{Q}^n_{2M},$$
$$Q^n_T \subseteq Q^n_{1T} \cap Q^n_{2M}$$

hold, and thus, every incidence vector of a tour satisfies (15), (16), (17), (18) and (23), (24), (25); and every incidence vector of a subset of a tour satisfies (12), (13), (14) and (20), (21), (22). We will see in Chapter 9 that it is possible to optimize over $Q^n_{1T} \cap Q^n_{2M}$ in polynomial time and that excellent lower bounds for the length of the shortest tour can be obtained this way.

A $\{0, 2\}$-*matching* is an assignment of the integers 0, 2 to the edges of a graph G such that for every vertex the sum of the integers on the incident edges is at most 2. The paper by Cornuéjols & Pulleyblank [1982] contains a study of the polytope $P(G)$ which is the convex hull of the $\{0, 2\}$-matchings and tours in G and its relation to Q^n_T. In fact, they show among other things that for n odd, Q^n_T is a face of $P(K_n)$ and for any facet F of Q^n_T there is a unique facet of $P(K_n)$ whose intersection with Q^n_T is exactly F. Recently, Hartvigsen [1984] investigated perfect 2-matchings without triangles and $\{0, 2\}$-matching without pentagons, but complete characterizations of the associated polytopes could not be obtained.

3.2 The asymmetric case

The most common relaxation of the asymmetric TSP is the assignment problem. An *assignment* B in a complete digraph $D_n = (V, A)$ is a set of arcs such that every vertex of D_n is the head and the tail of exactly one arc of B. In other words, B is a set of disjoint directed cycles in D_n such that every vertex is on a cycle. The *assignment polytope* P^n_A (on D_n) is the convex hull of all incidence vectors of assignments in D_n. There is of course a monotone version $\tilde{P}^n_A$ which is the convex hull of all incidence vectors of subsets of assignments in D_n and clearly we have $P^n_T \subseteq P^n_A$ and $\tilde{P}^n_T \subseteq \tilde{P}^n_A$. The following result has been proved (in different contexts) by various people and usually is associated with the names of Birkhoff and Von Neumann.

Theorem 5 *Let $n \geq 2$ and $D_n = (V, A)$ be the complete digraph on n vertices. Then*

$$P^n_A = \{x \in \mathbb{R}^A \mid x \text{ satisfies (7) and } x \geq 0\},$$

$$\tilde{P}^n_A = \{x \in \mathbb{R}^A \mid x \text{ satisfies (8) and } x \geq 0\}.$$

Note that in the assignment problem usually loops (i, i), $i = 1, \ldots, n$, are allowed. In other words, the set of assignments corresponds to the set of all permutations of the numbers $1, \ldots, n$. Since loops are of no interest for the TSP, we consider a slightly different definition; our assignments correspond to the set of permutations of $\{1, \ldots, n\}$ leaving no element fixed.

Theorem 5 is implied by the fact that the matrix corresponding to the equation system (7) is *totally unimodular*, i.e. every square submatrix has determinant +1, 0 or −1. It is nice to know that all vertices of the polyhedra defined by (7) (respectively (8)) and the nonnegativity conditions are integral, but we do not get any new polyhedral information for the asymmetric traveling salesman polytope from this fact.

It is easy to see that the rank of the $(2n, n^2 - n)$-matrix given by (7) is $2n - 1$ and that each row can be written as a linear combination of the

$2n-1$ other rows, for $n \geqslant 3$. This implies that $\dim(P^n_A) \leqslant |A| - 2n + 1$, and one can show the following.

Proposition 3 $\dim(P^n_A) = |A| - 2|V| + 1 = n^2 - 3n + 1$, $n \geqslant 3$.

We now want to consider a different approach which resembles the 1-tree relaxation in the symmetric case. To get a convenient description we have to reformulate the asymmetric TSP slightly.

Let $D_n = (V, A)$ be the complete digraph on the vertices $\{1, \ldots, n\}$. Denote by $D'_n = (V', A')$ the digraph on the vertices $\{1, \ldots, n, n+1\}$ defined in the following way

$$A' := \{(i,j) \mid i,j \in \{2, \ldots, n\}, i \neq j\} \cup \{(1,j) \mid j \in \{2, \ldots, n\}\} \cup \{(i, n+1) \mid i \in \{2, \ldots, n\}\}.$$

D'_n can be viewed as follows. We take D_n, add a new vertex $n+1$, take all those arcs in D_n going into vertex 1 and let them go into vertex $n+1$. Thus D'_n has the same number of arcs as D_n, and every tour in D_n corresponds to a directed Hamiltonian path in D'_n from 1 to $n+1$, and vice versa.

Let $\mathcal{I}_n$ be the set of all directed Hamiltonian paths from 1 to $n+1$ in D'_n and $\tilde{\mathcal{I}}_n$ be its monotonization. Then it is easy to see that $\mathcal{I}_n = \mathcal{T}_n$ and $\tilde{\mathcal{I}}_n = \tilde{\mathcal{T}}_n$ (with the obvious reinterpretation of arcs) and hence that $P^n_T (\tilde{P}^n_T)$ is the convex hull of all incidence vectors of (subsets of) directed Hamiltonian paths from 1 to $n+1$ in D'_n.

Let us define the following independence systems on A':

$$\vec{\mathcal{I}}_n := \{B \subseteq A' \mid |B \cap \vec{\delta}(v)| \leqslant 1 \text{ for all } v \in \{1, \ldots, n\}\},$$
$$\overleftarrow{\mathcal{I}}_n := \{B \subseteq A' \mid |B \cap \overleftarrow{\delta}(v)| \leqslant 1 \text{ for all } v \in \{2, \ldots, n+1\}\},$$
$$\mathcal{I}^F_n := \{B \subseteq A' \mid B \text{ contains no cycle (in the undirected sense)}\}.$$

It is well known and an easy exercise to show that $(A', \vec{\mathcal{I}}_n)$, $(A', \overleftarrow{\mathcal{I}}_n)$ and $(A', \mathcal{I}^F_n)$ are matroids on A'. Moreover, we have the following result.

Lemma 4 *$\mathcal{I}_n = \vec{\mathcal{I}}_n \cap \overleftarrow{\mathcal{I}}_n \cup \mathcal{I}^F_n$ and $\mathcal{I}_n$ is the intersection of the bases (of A') of $\vec{\mathcal{I}}_n$, $\overleftarrow{\mathcal{I}}_n$ and $\mathcal{I}^F_n$.*

Proof Left as an exercise for the reader.

Lemma 4 justifies the statement that the asymmetric can be viewed as the intersection of three matroids.

We know from a result of Edmonds [1970] how a linear description of the polytope associated with the intersection of two matroids can be obtained. (This result does not extend to the intersection of three or more matroids!) Thus we are able to give complete and nonredundant characterizations of the polytopes corresponding to the independence systems $\vec{\mathcal{I}}_n \cap \overleftarrow{\mathcal{I}}_n$, $\vec{\mathcal{I}}_n \cap \mathcal{I}^F_n$, $\overleftarrow{\mathcal{I}}_n \cap \mathcal{I}^F_n$ (respectively their bases). The last two have been investigated in detail [Giles, 1975; Grötschel, 1977a].

As a matter of fact, the polytope associated with $\vec{\mathcal{I}}_n \cap \overleftarrow{\mathcal{I}}_n$ is the polytope

$\tilde{P}^n_A$; the polytope associated with the bases (of A') of $\vec{\mathscr{I}}_n \cap \overleftarrow{\mathscr{I}}_n$ is the assignment polytope P^n_A.

An arc set in A' containing no cycle (in the undirected sense) and in which every vertex is the head (tail) of at most one arc is called a *branching* (*antibranching*). A branching (antibranching) in D'_n with n arcs is called *arborescence* (*antiarborescence*). Thus $\vec{\mathscr{I}}_n \cap \mathscr{I}^F_n$ ($\overleftarrow{\mathscr{I}}_n \cap \mathscr{I}^F_n$) is the set of branchings (antibranchings) in D'_n, and the bases (of A') of $\vec{\mathscr{I}}_n \cap \mathscr{I}^F_n$ ($\overleftarrow{\mathscr{I}}_n \cap \mathscr{I}^F_n$) are exactly the arborescences (antiarborescences) in D'_n. Let $P^n_{\vec{B}}$ ($P^n_{\overleftarrow{B}}$) denote the convex hull of the incidence vectors of the arborescences (antiarborescences) in D'_n, and $\tilde{P}^n_{\vec{B}}$ ($\tilde{P}^n_{\overleftarrow{B}}$) the convex hull of the incidence vectors of branchings (antibranchings) in D'_n. We clearly have $P^n_T \subseteq P^n_{\vec{B}} \cap P^n_{\overleftarrow{B}}$ and $\tilde{P}^n_T \subseteq \tilde{P}^n_{\vec{B}} \cap \tilde{P}^n_{\overleftarrow{B}}$. These polytopes can be described as follows.

Theorem 6

$$\tilde{P}^n_{\vec{B}} = \{x \in \mathbb{R}^{A'} \mid x_{ij} \geq 0 \quad \textit{for all } (i,j) \in A', \tag{26}$$

$$x(\overleftarrow{\delta}(v)) \leq 1 \quad \textit{for all } v \in \{2, \ldots, n+1\}, \tag{27}$$

$$x(A(W)) \leq |W| - 1 \quad \textit{for all } W \subseteq V', |W| \geq 2, \; 1, n+1 \notin W\}, \tag{28}$$

$$\tilde{P}^n_{\overleftarrow{B}} = \{x \in \mathbb{R}^{A'} \mid x(\vec{\delta}(v)) \leq 1 \;\; \textit{for all } v \in \{1, \ldots, n\} \;\; \textit{and } x \textit{ satisfies } (26) \textit{ and } (28)\}, \tag{29}$$

$$P^n_{\vec{B}} = \{x \in \mathbb{R}^{A'} \mid x(\overleftarrow{\delta}(v)) = 1 \;\; \textit{for all } v \in \{2, \ldots, n+1\} \;\; \textit{and } x \textit{ satisfies } (26) \textit{ and } (28)\}, \tag{30}$$

$$P^n_{\overleftarrow{B}} = \{x \in \mathbb{R}^{A'} \mid x(\vec{\delta}(v)) = 1 \;\; \textit{for all } v \in \{1, \ldots, n\} \;\; \textit{and } x \textit{ satisfies } (26) \textit{ and } (28)\}. \tag{31}$$

Moreover, all these linear descriptions are nonredundant.

Note that (28) are just subtour elimination constraints, so we did not really get any new polyhedral information about the TSP. But on the other hand, it is interesting to see that certain degree constraints plus certain subtour elimination constraints define integral polyhedra. (The reinterpretation of the polyhedra $\tilde{P}_{\vec{B}}$, $\tilde{P}_{\overleftarrow{B}}$, etc., with respect to the complete digraph D_n is obvious.)

Since the assignment, branching and antibranching problem are solvable in polynomial time, we can optimize over $P^n_{\vec{B}} \cap P^n_{\overleftarrow{B}}$ and $\tilde{P}^n_{\vec{B}} \cap \tilde{P}^n_{\overleftarrow{B}}$ in polynomial time; cf. Chapter 9.

Exercises

6. Prove Lemma 3. (*Hint*: see Grötschel & Pulleyblank [1985] for the 'only if' part.)
7. Find complete and nonredundant systems of equations and inequalities

describing Q^3_{2M} and Q^4_{2M}. (Note that for $n = 3, 4$ the system (23), (24), (25) is complete for Q^n_{2M}, but redundant.)

8. It is also quite easy to describe the vertices of the polytope

$$\bar{Q}^n_{2M} := \{x \in \mathbb{R}^E \mid x \text{ satisfies (23) and (24)}\}.$$

Prove that the vertices of $\bar{Q}^n_{2M}$ are either incidence vectors of perfect 2-matchings or fractional vertices whose components are 0, 1 or $\frac{1}{2}$. Moreover, show that the edges corresponding to $\frac{1}{2}$-components of fractional vertices of $\bar{Q}^n_{2M}$ form an even number of disjoint cycles of odd length. (*Hint*: see Grötschel [1977a].)

9. Prove Lemma 4.

10. Find a representation of the symmetric TSP (i.e. of $\tilde{\mathcal{S}}_n$) as the intersection of matroids. (As a research problem, try to find the minimal number of matroids that is sufficient.)

4 THE SYMMETRIC TRAVELING SALESMAN POLYTOPES

We shall now study those properties of the traveling salesman polytopes $\tilde{Q}^n_{\mathrm{T}}$ and Q^n_{T} which cannot be derived from general results about combinatorial polyhedra or from the relaxations introduced in Section 3.1. In particular, we shall introduce all classes of facets of Q^n_{T} and $\tilde{Q}^n_{\mathrm{T}}$ which are known at present (to our knowledge).

The results are not presented in chronological order, rather in an order that minimizes space. The presentation is based on several papers [Grötschel, 1977a, 1980a; Grötschel & Padberg, 1974, 1975a, 1978, 1979a, 1979b; Grötschel & Pulleyblank, 1985; Maurras, 1975, 1976; Papadimitriou & Yannakakis, 1984], from which most of the results are taken. Research on Q^n_{T} (and P^n_{T}) was also very active in the mid-1950s. A summary of these developments is given by Gomory [1966] and Grötschel [1977a].

4.1 Some properties of Q^n_{T}, relations to $\tilde{Q}^n_{\mathrm{T}}$

We shall first investigate the dimension of Q^n_{T} and the trivial inequalities. Then we shall study some interesting relations between Q^n_{T} and $\tilde{Q}^n_{\mathrm{T}}$ and state a general procedure by which, from each inequality defining a facet of Q^n_{T}, an inequality defining a facet of $\tilde{Q}^n_{\mathrm{T}}$ can be obtained.

From Theorem 4 we know that $\dim(Q^n_{2M}) = |E| - n$, hence $\dim(Q^n_{\mathrm{T}}) \leqslant |E| - n$ since $Q^n_{\mathrm{T}} \subseteq Q^n_{2M}$. We shall now prove that the dimensions of Q^n_{2M} and Q^n_{T} are equal. To give applications of the two proof techniques introduced in Section 1 (cf. the discussion after Theorem 1), we outline two proofs, each based on a different technique.

Theorem 7 *The dimension of Q^n_{T} equals*

$$d_n := |E| - |V| = \tfrac{1}{2}n(n-3) \qquad \textit{for all } n \geqslant 3.$$

First proof [Grötschel & Padberg, 1975a, 1979a] We first have to state a well known graph-theoretical lemma.

Lemma 5 *Let $K_n = (V, E)$ be the complete graph on n vertices, and let k denote any integer.*
(i) *If $|V| = 2k+1$, then there exist k edge-disjoint tours T_i such that $E = \bigcup_{i=1}^{k} T_i$.*
(ii) *If $|V| = 2k$, then there exist $k-1$ edge-disjoint tours T_i and a perfect 1-matching M edge-disjoint from any T_i such that $E = M \cup \bigcup_{i=1}^{k-1} T_i$.*

We now show that Q_T^n contains $d_n + 1$ linearly independent tours. For $n = 3$ the claim is trivially true.

(a) Assume that $n = 2k+2$ with $k \geq 1$ and integer. The subgraph $K_{n-1} = (V', E')$ of K_n induced by the $n-1$ first vertices is again complete and by the lemma, its edge set is the union of k edge-disjoint tours τ_i of length $n-1$. From each $(n-1)$-tour τ_i we construct $n-1$ tours τ_{ij} of length n by replacing, one at a time, each edge $\{u, v\} \in \tau_i$ by the path $[u, n, v]$. The incidence matrix of the tours τ_{ij} for $j = 1, \ldots, n-1$ (rows) versus the edges of K_n (columns) contains the submatrix $E - I$, where E is the $(n-1)\times(n-1)$ matrix of all ones and I is the $(n-1)\times(n-1)$ identity matrix. Furthermore, we obtain a total of $k(n-1) = d_n + 1$ tours. Since the tours τ_i of length $n-1$ are edge-disjoint, the incidence matrix of all $d_n + 1$ tours τ_{ij} contains a $(d_n+1)\times(d_n+1)$ submatrix N which is block-diagonal and whose diagonal blocks are all equal to $E - I$ after a suitable arrangement of the rows and columns. Since $E - I$ is nonsingular, it follows that N is nonsingular and consequently, Theorem 7 holds if $n = 2k+2$.

(b) Assume that $n = 2k+1$ with $k \geq 2$ and integer. We proceed as in the case (a) and construct $(k-1)(n-1)$ linearly independent tours from the $k-1$ tours of length $n-1$. The perfect matching in K_{n-1} is completed arbitrarily to an $(n-1)$-tour in K_{n-1} and subsequently used to construct k tours of length n by replacing, one at a time, each edge $\{u, v\} \in M$ by the path $[u, n, v]$. In this way we obtain a total of $d_n + 1$ tours whose incidence matrix contains a $(d_n+1)\times(d_n+1)$ block-triangular matrix N with $k-1$ blocks $E - I$ of size $(n-1)\times(n-1)$ and an additional block $E' - I'$ of size $k \times k$. □

Second proof [Maurras, 1975] We know that $Q_T^n \subseteq \{x \in \mathbb{R}^E \mid x(\delta(i)) = 2, i = 1, \ldots, n\}$ and that the matrix corresponding to these n equations has rank n. To prove our claim, we have to show that for every hyperplane $H = \{x \mid a^T x = a_0\}$ containing Q_T^n, the normal vector a is a linear combination of the normal vectors of the known equation system. So suppose that $a^T x = a_0$, $a \neq 0$, is an equation satisfied by all $x \in Q_T^n$. We have to prove that there are λ_i, $i = 1, \ldots, n$, with $a^T x = \sum_{i=1}^{n} \lambda_i x(\delta(i))$.

The incidence vectors $x^{\delta(i)}$ have a very special structure which we will utilize. By adding appropriate multiples of the incidence vectors of $\delta(1)$, $\delta(2)$ and $\delta(3)$ to twice the negative of a, we get a vector which is 0 on the triangle {1, 2}, {2, 3}, {1, 3}. Namely, setting

$$\lambda_1 := a_{12} + a_{13} - a_{23},$$
$$\lambda_2 := a_{12} + a_{23} - a_{13},$$
$$\lambda_3 := a_{13} + a_{23} - a_{12}$$

and

$$a' := \lambda_1 x^{\delta(1)} + \lambda_2 x^{\delta(2)} + \lambda_3 x^{\delta(3)} - 2a,$$

we obtain $a'_{12} = a'_{13} = a'_{23} = 0$. Now we set $\lambda_i := -a'_{1i}$, $i = 4, \ldots, n$, and define $b := a' + \sum_{i=4}^{n} \lambda_i x^{\delta(i)}$. The definition clearly implies that $b_{1i} = 0$, $i = 2, \ldots, n$, and that $b_{23} = 0$. The construction above works because of the following reason. One can show that the (n, n)-matrix M having the n vectors $x^{\delta(i)}$ as rows and consisting of the columns corresponding to the edges $S := \{\{1, i\} \mid i = 2, \ldots, n\} \cup \{\{2, 3\}\}$ is nonsingular. The λ_i constructed above are the unique solutions of $M^T x = 2\bar{a}$, where $\bar{a}$ arises from a by deleting all components except for those belonging to S. (For further details, see the remarks after Theorem 9, and for another application of this technique, see the proof of Theorem 11.)

We have now used up our degrees of freedom. If we can show that the vector b constructed above equals 0 then we are done, since in this case,

$$a = \tfrac{1}{2} \sum_{i=1}^{n} \lambda_i x^{\delta(i)}.$$

But as we shall see, this is quite easy.

Let $i \in \{4, \ldots, n\}$ be any vertex and P_{i3} be any path from i to 3 going through all vertices in $\{3, \ldots, n\}$. Define the following tours: $\tau_1 := P_{i3} \cup \{\{1, i\}, \{1, 2\}, \{2, 3\}\}$ and $\tau_2 := P_{i3} \cup \{\{2, i\}, \{1, 2\}, \{1, 3\}\}$. If x^{τ_i} denotes the incidence vector of τ_i, then we have $(x^{\delta(j)})^T x^{\tau_i} = 2$ and $a^T x^{\tau_i} = a_0$. Since b is a linear combination of the vectors $x^{\delta(i)}$ and a, $b^T x^{\tau_i} = b_0$ for some constant b_0. Therefore, $0 = b^T x^{\tau_1} - b^T x^{\tau_2} = b_{1i} + b_{23} - b_{2i} - b_{13} = -b_{2i}$. This proves that $b_{2i} = 0$, $i = 1, 3, 4, \ldots, n$.

Similarly, we obtain $b_{3i} = 0$ for all $i \neq 3$, and by iterating this procedure we get $b_{ij} = 0$ for all $i \neq j$, which proves our claim. □

The proof that the trivial inequalities $0 \leq x_{ij} \leq 1$ define facets of the monotone traveling salesman polytope $\tilde{Q}_T^n$ is trivial. The proof that these inequalities define facets of Q_T^n requires about the same amount of technical detail as the proof of Theorem 7. This illustrates also that lower-dimensional polyhedra are not as easy to handle as full-dimensional ones and why it is preferable (from a technical point of view) to deal with full-dimensional polyhedra.

Theorem 8 *Let $K_n = (V, E)$ be the complete graph on n vertices.*
(a) *The inequalities $x_{ij} \leq 1, \{i, j\} \in E$, define facets of Q_T^n for all $n \geq 4$.*
(b) *The inequalities $x_{ij} \geq 0, \{i, j\} \in E$, define facets of Q_T^n for all $n \geq 5$.*

Since it is somewhat easier to deal with $\tilde{Q}_T^n$ than with Q_T^n it would be nice to have a theorem which characterizes those facet-defining inequalities of $\tilde{Q}_T^n$ which also define facets of Q_T^n. Clearly, not all of the inequalities defining facets of $\tilde{Q}_T^n$ have this property. We shall see later that the degree constraints (6) define facets of $\tilde{Q}_T^n$, but they are satisfied with equality for all $x \in Q_T^n$. By comparing Corollary 2(a) and Proposition 2 with Theorem 8(b) and (a), one can see that there are even slight differences with respect to the trivial inequalities. This could be explained as an irregularity of low dimensions, but it has to be taken into account. Anyway, a reasonable result of the desired type is not known. Let us formulate the problem in a somewhat more modest form.

Research problem *Find (reasonable) sufficient conditions which imply that an inequality defining a facet for $\tilde{Q}_T^n$ also defines a facet for Q_T^n.*

Since Q_T^n is a face of $\tilde{Q}_T^n$, we know that for every facet of Q_T^n there is at least one inequality defining it which also defines a facet of $\tilde{Q}_T^n$. But since we can add degree equations (5) to every facet-defining inequality of Q_T^n there are many inequalities defining a facet of Q_T^n which are not even valid with respect to $\tilde{Q}_T^n$. However, Grötschel & Pulleyblank [1985] describe a procedure by means of which every facet-defining inequality $a^Tx \leq a_0$ of Q_T^n can be turned into an inequality which is equivalent to $a^Tx \leq a_0$ with respect to Q_T^n and which defines a facet of $\tilde{Q}_T^n$. We shall briefly describe this method here.

For notational convenience, let us write

$$Ax = 2 \tag{32}$$

for the degree equations $x(\delta(i)) = 2$, $i = 1, \ldots, n$, and suppose that $a^Tx \leq a_0$ defines a facet of Q_T^n.

If $a \geq 0$ holds, then $a^Tx \leq a_0$ is valid for $\tilde{Q}_T^n$. (Why?) If a has negative coefficients, then we choose a $\lambda \in \mathbb{R}^n$ with sufficiently large coefficients (it should be clear how these have to be selected) such that $\bar{a}^T := a^T + \lambda^T A \geq 0$. Let $\bar{a}_0 := a_0 + \lambda^T 2$, then $\bar{a}^Tx \leq \bar{a}_0$ is equivalent to $a^Tx \leq a_0$ with respect to Q_T^n and valid for $\tilde{Q}_T^n$.

Let us assume therefore that the initial inequality $a^Tx \leq a_0$ satisfies $a \geq 0$, and let $E^0(a)$ be the set of edges in K_n corresponding to a zero coefficient of a, i.e. $E^0(a) = \{\{i, j\} \in E \mid a_{ij} = 0\}$. Denote by $A_{E^0(a)}$ the submatrix of A (cf. (32)) consisting of all rows of A and the columns corresponding to the edges in $E^0(a)$. We then carry out the following computations in sequence.

Step 1. If the rows of $A_{E^0(a)}$ are linearly independent, go to Step 2. If not, find a $\lambda \in \mathbb{R}^n$ such that $\lambda^T A \neq 0$ and $\lambda^T A_{E^0(a)} = 0$ (clearly, such a λ exists).

Step 2. Set $\bar{a}^T := a^T - \mu\lambda^T A$, $\bar{a}_0 := a_0 - \mu\lambda^T 2$ where μ is an appropriately chosen number such that $\bar{a} \geq 0$ and $E^0(a) \subset E^0(\bar{a})$. Set $a := \bar{a}$, $a_0 := \bar{a}_0$ and go to Step 1.

Let us call the above procedure *reducing* the inequality $a^Tx \leq a_0$. It is clear that after at most $|E|$ iterations we obtain an inequality, say $ax \leq a_0$, such that $A_{E^0(a)}$ has rank n, and $a \geq 0$. To normalize our procedure we add a final step.

Step 3. Scale $a^T x \leq a_0$ such that the smallest nonzero coefficient of a has value 1, and stop.

We call an inequality obtained by applying the above algorithm *support reduced*. One could also say that it is a normalized inequality with minimal support. Before stating the main result about the algorithm, we have to exclude a trivial case.

Lemma 6 *Let $a^T x \leq a_0$ be a support-reduced facet-inducing inequality for Q_T^n. Then $a^T x \leq a_0$ induces a trivial facet of Q_T^n if and only if*

(a) *$E - E^0(a) = \delta(v) - \{k\}$ for some $v \in V$ and $k \in \delta(v)$, and in this case $a_{ij} = 1$ for all $\{i, j\} \in \delta(v) - \{k\}$ and $a_0 = 2$; or*

(b) *$E^0(a) = \delta(v) \cup \{k\}$ for some $v \in V$ and $k \in E(V - \{v\})$, and in this case $a_{ij} = 1$ for all $\{i, j\} \in E - E^0(a)$ and $a_0 = n - 2$.*

The following result was proved by Grötschel & Pulleyblank [1985].

Theorem 9 *Let $a^T x \leq a_0$ be a support-reduced facet-inducing inequality for Q_T^n not of the form* (a) *or* (b) *of Lemma 6. Then $a^T x \leq a_0$ is facet-inducing for $\bar{Q}_T^n$.*

As we shall see later, Theorem 9 will be quite helpful for obtaining facet-defining inequalities for $\bar{Q}_T^n$.

A very important point is that the linear independence of the rows of $A_{E^0(a)}$ can be checked easily using graph-theoretical (or matroidal) methods. More precisely, the rows of $A_{E^0(a)}$ are linearly independent if and only if the edge set $E^0(a)$ is a basis of the real (matric) matroid $\mathcal{M}(K_n)$ of K_n, $n \geq 3$; and $E^0(a)$ is a basis of $\mathcal{M}(K_n)$ if and only if it is a maximal subset of E such that each component of $E^0(a)$ contains exactly one cycle of odd length and no cycle of even length; for details, see Grötschel & Pulleyblank [1985].

4.2 Clique tree inequalities

In this section we investigate a class of inequalities introduced by Grötschel & Pulleyblank [1985] that subsumes all the nontrivial inequalities valid for $\bar{Q}_T^n$ introduced so far, and also subsumes the classes of inequalities studied by Chvátal [1973a] and Grötschel & Padberg [1979a, 1979b].

A *clique* in a graph $G = (V, E)$ is a set C of vertices such that any two vertices in C are adjacent and such that C is maximal with respect to this property. A set A of vertices in a connected graph G is called an *articulation set* if the graph obtained from G by removing A is disconnected.

A *clique tree* is a connected graph C composed of cliques which satisfy the following properties (in the following we shall always consider clique trees as subgraphs of K_n):

(1) The cliques are partitioned into two sets, the set of *handles* and the set of *teeth*.
(2) No two teeth intersect.
(3) No two handles intersect.
(4) Each tooth contains at least two and at most $n-2$ vertices and at least one vertex not belonging to any handle.
(5) The number of teeth that each handle intersects is odd and at least three.
(6) If a tooth T and a handle H have a nonempty intersection, then $H \cap T$ is an articulation set of the clique tree.

Figure 8.1 shows an example of a clique tree, where cliques are indicated by ellipse-shaped figures. Each ellipse containing a '*' is a tooth. The '*' indicates that there must be a vertex in the respective tooth which does not belong to any handle.

We call a clique tree *simple* if any handle and any tooth have at most one vertex in common.

Suppose we have a clique tree C with handles $H_1, H_2, \ldots, H_r$ and teeth $T_1, T_2, \ldots, T_s$. We show below that the following *clique tree inequality* is valid for Q_T^n and $\tilde{Q}_T^n$ (in fact it defines a facet):

$$\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) \leq \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \frac{s+1}{2} = s(C), \quad (33)$$

where for every tooth T_j the integer t_j denotes the number of handles which intersect T_j, $j = 1, \ldots, s$. The right-hand side $s(C)$ of (33) is called the *size* of C.

Note that in the case where there is a tooth T and a handle H with $|H \cap T| \geq 2$, the coefficients on the left-hand side of (33) are 0, 1 and 2. The inequality (33) is a 0–1 inequality only if the clique tree is simple. If W is the set of all vertices of a clique tree, then, for simple clique trees, inequality

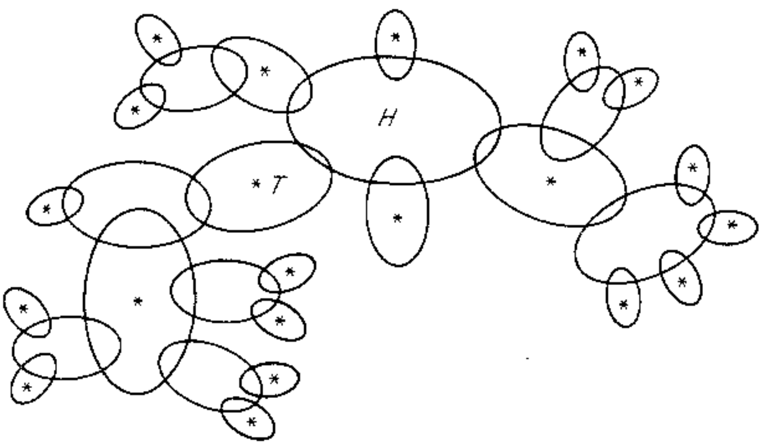


Figure 8.1

(33) can be written as

$$\sum_{i=1}^{r} x(E(H_i))+\sum_{j=1}^{s} x(E(T_j))\leq |W|-\frac{s+1}{2}.$$

We shall now consider various special cases of (33). A clique tree with only one handle H is called a *comb*, and the corresponding inequality

$$x(E(H))+\sum_{j=1}^{s} x(E(T_j))\leq |H|+\sum_{j=1}^{s} (|T_j|-1)-\frac{s+1}{2} \tag{34}$$

is called a *comb inequality*. These inequalities were introduced and studied by Grötschel & Padberg [1979a, 1979b].

Comb inequalities in turn are generalizations of inequalities introduced by Chvátal [1973a]. A comb is a *Chvátal* comb if every tooth has exactly one vertex in common with the handle, i.e. Chvátal combs are simple combs. The Chvátal comb inequality is the same as (34). Actually, Chvátal introduced a slightly larger class of inequalities (cf. Chapter 11), but all those not contained in our definition can be shown to be redundant.

The class of Chvátal combs generalizes Edmonds' 2-matching inequalities (22). Namely, the 2-matching inequalities (29) (except for those with $k=1$) are exactly those Chvátal comb inequalities where every tooth contains exactly two vertices.

In addition, the class of clique tree inequalities also contains the subtour elimination constraints (10), except for $|W|=n-1$. They are exactly those clique tree inequalities having a clique tree consisting of one tooth and no handle. Thus, in particular, the trivial inequalities $x_{ij}\leq 1$ are special clique tree inequalities.

We now want to prove that the clique tree inequalities (33) are valid for $\tilde{Q}^n_T$ (and thus for Q^n_T). For this we introduce the following two splitting operations.

(a) ***Splitting a clique tree at a tooth and a handle.*** *Let C be a clique tree and T a tooth of C. Let H be a handle of C intersecting T. Delete the vertices $H-T$ from C and let C'' be the component of $C-(H-T)$ containing T. Delete all vertices from C which are in handles meeting T but not in T or H. Let C' be the component of this graph containing T. Then C' and C'' are clique trees called the clique trees obtained from C by splitting at T and H.*

(b) ***Splitting a clique tree at a handle.*** *Let C be a clique tree and H a handle of C. Let $T_1,\ldots,T_k$ be the teeth of C which intersect H. For every tooth T_i, $i\in\{1,\ldots,k\}$, let C_i be the clique tree not containing H obtained from C by splitting at T_i and H. Then the clique trees $C_1,\ldots,C_k$ are called the clique trees obtained from C by splitting at H.*

Figures 8.2(a) and (b) show the operations (a) and (b), respectively, applied to the clique tree given in Figure 8.1, where H and T indicate the relevant tooth and handle.

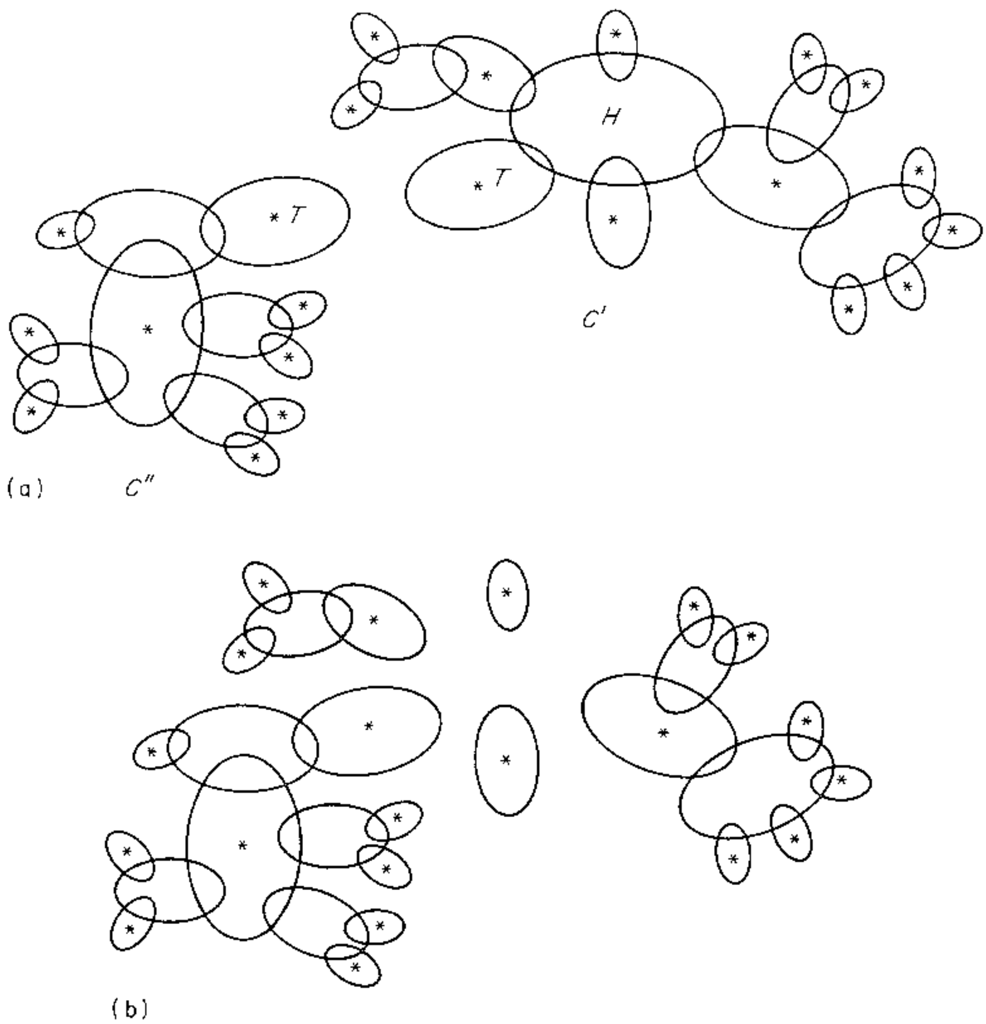


Figure 8.2

The following observation is immediate.

Lemma 7

(a) *Let C' and C'' be the clique trees obtained from C by splitting at tooth T and handle H (as in* (a) *above). Then*

$$s(C') + s(C'') = s(C) + |T| - 1.$$

(b) *Let C be a clique tree and H a handle of C intersecting k teeth. Let $C_1, \ldots, C_k$ be the clique trees obtained from C by splitting at handle H (as in* (b) *above). Then*

$$\sum_{i=1}^{k} s(C_i) = s(C) - |H| + \frac{k+1}{2}.$$

The proof of the validity of (33) is given by Grötschel & Pulleyblank [1985]. It is inductive and uses the fact that the subtour elimination constraints (10) are valid.

Theorem 10 *Let C be a clique tree in K_n with handles $H_1, \ldots, H_r$ and teeth $T_1, \ldots, T_s$. Then the clique tree inequality*

$$\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) \leqslant \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \frac{s+1}{2} = s(C)$$

is valid with respect to $\tilde{Q}_T^n$ (and hence with respect to Q_T^n).

Proof We prove the theorem by induction on the number of handles. If C has no handle, then the clique tree consists of just one tooth, and the clique tree inequality is a subtour elimination constraint. Thus there is nothing to prove.

Suppose the claim is true for all clique trees with r handles, and assume C is a clique tree with $r+1$ handles. Pick any handle H of C. Let $T_1, \ldots, T_k$ be the teeth of C intersecting H, and let $C_1, \ldots, C_k$ be the clique trees obtained from C by splitting at H. Every such clique tree has at most r handles. By construction C_i contains T_i, $i = 1, \ldots, k$. Let $a_i^T x \leqslant s(C_i)$ be the corresponding clique tree inequalities.

For every clique tree C_i, $i \in \{1, \ldots, k\}$, let $\bar{C}_i$ be the clique tree obtained from C_i after replacing T_i by $T_i - H$, and let $\bar{a}_i^T x \leqslant s(\bar{C}_i)$ be the corresponding clique tree inequality. (If a tooth T_i contains only two vertices, then $\bar{C}_i$ is not a clique tree, but the counting arguments used below remain valid.)

By Lemma 7 we have

$$\sum_{i=1}^{k} s(C_i) = s(C) - |H| + \frac{k+1}{2}$$

which implies

$$\sum_{i=1}^{k} s(\bar{C}_i) = s(C) - |H| - \sum_{i=1}^{k} |H \cap T_i| + \frac{k+1}{2}.$$

From this we obtain

$$\begin{aligned} 2\left(\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j))\right) &\leqslant \sum_{i=1}^{k} (a_i^T x + \bar{a}_i^T x + x(E(H \cap T_i))) + \sum_{v \in H} x(\delta(v)) \\ &\leqslant \sum_{i=1}^{k} (s(C_i) + s(\bar{C}_i) + |H \cap T_i| - 1) + 2|H| \\ &= 2s(C) + 1. \end{aligned}$$

For every incidence vector of a subset of a tour, the left-hand side above is an even integer. So, dividing by 2 and rounding down the right-hand side we get the desired result. □

4.3 'Nice' facets of Q_T^n and $\tilde{Q}_T^n$

The clique tree inequalities (33) and all their special cases are in a certain intuitive sense 'nice', since they can be described easily by formulas, we have easy-to-state inductive definitions; for some special cases, good separation algorithms are known (cf. Chapter 9), and there is some hope that they may be handled efficiently in cutting plane procedures. (For some other 'bad' inequalities to be discussed in the next section no such hope exists at present.)

In this section we prove that subtour elimination constraints define facets of Q^n_T, and we indicate how to show that clique tree inequalities define facets.

Theorem 11 *Let $n \geqslant 4$ and W be a vertex set in $K_n = (V, E)$ with $2 \leqslant |W| \leqslant n-2$. Then the subtour elimination constraint $x(E(W)) \leqslant |W| - 1$ defines a facet of Q^n_T.*

Proof Let us first assume that $n \geqslant 6$ and that $W = \{1, 2, \ldots, k\}$, $3 \leqslant k \leqslant n-3$. For notational convenience we denote $x(E(W)) \leqslant |W| - 1$ by $a^T x \leqslant a_0$.

Suppose now that $b^T x \leqslant b_0$ is a valid inequality for Q^n_T satisfying $\{x \in Q^n_T \mid a^T x = a_0\} \subseteq \{x \in Q^n_T \mid b^T x = b_0\}$. If we can show that, for some $\alpha \geqslant 0$ and $\lambda \in \mathbb{R}^n$, $b^T = \alpha a^T + \lambda^T A$ then we are done by Theorem 1(d) (matrix A is defined in (32)).

The edge set $F := \{\{1, i\}, i = 2, \ldots, n\} \cup \{\{2, 3\}\}$ contains a spanning tree and one odd cycle but no even cycle, thus it is a basis of $\mathcal{M}(K_n)$, which means that the matrix A_F consisting of all rows of A and the columns corresponding to edges in F is nonsingular. Thus there exists a vector $\bar{\lambda} \in \mathbb{R}^n$ such that $\bar{b}^T := b^T + \bar{\lambda}^T A$ satisfies $\bar{b}_{1i} = a_{1i}$, $i = 2, \ldots, k$ and $\bar{b}_{23} = a_{23}$. Recalling that a is the incidence vector of $E(\{1, \ldots, k\})$, we may therefore assume that our initial vector b satisfies

$$\begin{aligned} b_{1i} &= 1\ (= a_{1i}), \qquad i = 2, \ldots, k, \\ b_{1i} &= 0\ (= a_{1i}), \qquad i = k+1, \ldots, n, \\ b_{23} &= 1\ (= a_{23}). \end{aligned}$$

Let $i \in \{4, \ldots, k\}$, for convenience say $i = k$, and consider the tours $\tau_1 = \langle 1, k, k-1, \ldots, 4, 2, 3, k+1, \ldots, n \rangle$ and $\tau_2 = \langle 1, 2, 4, \ldots, k-1, k, 3, k+1, \ldots, n \rangle$, then the incidence vectors x^{τ_1} and x^{τ_2} of τ_1 and τ_2 satisfy $a^T x = a_0$ and hence $b^T x = b_0$ with equality. This implies $0 = b_0 - b_0 = b^T x^{\tau_1} - b^T x^{\tau_2} = b_{1k} + b_{23} - b_{12} - b_{3k} = 1 - b_{3k}$. Thus $b_{3k} = 1$. By iterating this argument we obtain

$$b_{ij} = 1, \qquad 1 \leqslant i < j \leqslant k.$$

Let $i \in \{k+1, \ldots, n\}$; say $i = n$, and consider the tours $\tau_3 = \langle 1, 2, \ldots, n \rangle$ and $\tau_4 = \langle 2, 1, 3, 4, \ldots, n \rangle$ whose incidence vectors satisfy $a^T x = a_0$ and consequently $b^T x = b_0$ as well. We obtain $0 = b^T x^{\tau_3} - b^T x^{\tau_4} = b_{23} + b_{1n} - b_{13} - b_{2n} = -b_{2n}$. Applying this construction repeatedly we obtain

$$b_{ij} = 0, \qquad i \in \{1, \ldots, k\}, \qquad j \in \{k+1, \ldots, n\}.$$

Using a similar construction one can also show that there exists a number β

such that

$$b_{ij} = \beta, \qquad k+1 \leq i < j \leq n.$$

This implies that $b^T x = x(E(W)) + \beta x(E(V-W))$ and that $b_0 = |W| - 1 + \beta(|V-W|-1)$. The incidence vector of the tour $\tau_5 = \langle 1, n, 2, \ldots, k, k+1, \ldots, n-1 \rangle$ satisfies $b^T x^{\tau_5} = |W| - 2 + \beta(|V-W|-2)$. Since $b^T x \leq b_0$ is valid for Q_T^n we also have $b^T x^{\tau_5} \leq |W| - 1 + \beta(|V-W|-1)$; hence $\beta \geq -1$ has to hold. From this we obtain the desired representation of b as

$$b^T = \alpha a^T + \lambda^T A$$

where $\alpha = 1 + \beta$, $\lambda_i = -\beta/2$ for $i = 1, \ldots, k$ and $\lambda_i = \beta/2$ for $i = k+1, \ldots, n$.

The case $|W| = 2$ follows from Theorem 8(a). The case $|W| = n-2$ follows from Theorem 8(a) combined with Lemma 8(a). The cases $n = 4, 5$ are easy. □

The proof given above implicitly contains part (a) of the following lemma.

Lemma 8 *Let $n \geq 4$.*

(a) *Suppose W and W' are different vertex sets in K_n with at least two and at most $n-2$ vertices. Then the corresponding subtour elimination constraints are equivalent with respect to Q_T^n if and only if $W' = V - W$.*

(b) *Let $\delta(W)$, $W \subseteq V$, denote the set of edges in K_n with one endpoint in W and the other in $V-W$. For every $W \subseteq V$, $2 \leq |W| \leq n-2$, the so-called loop constraint*

$$x(\delta(W)) \geq 2$$

is equivalent to $x(E(W)) \leq |W| - 1$ with respect to Q_T^n (and therefore defines a facet of Q_T^n).

The only existing proof that clique tree inequalities define facets of Q_T^n is quite involved. We give here an outline only of the basic proof technique. Grötschel & Padberg [1979a] have shown (by brute force) that the 2-matching inequality derived from the smallest comb (a comb with a handle of cardinality 3 and three teeth of cardinality 2) defines a facet of Q_T^n for all $n \geq 6$. Such a smallest comb is shown in Figure 8.3. The comb (or

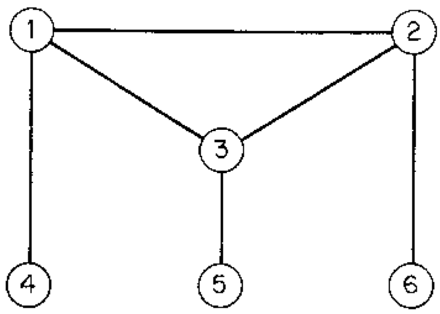


Figure 8.3

2-matching) inequality for the comb of Figure 8.3 is

$$x_{12}+x_{13}+x_{14}+x_{23}+x_{26}+x_{35}\leqslant 4.$$

Grötschel & Padberg [1979b] have established various lifting theorems which show how one can blow up a comb by adding new teeth, or enlarging a handle or a tooth such that the new comb inequality is a facet in case the original one was a facet. (In fact, the lifting theorems are more general.) Using the result that the smallest comb induces a facet, one can then derive that all comb inequalities define facets of Q_T^n. For completeness we state two examples of such lifting theorems. In these theorems $K_n=(V,E)$ is the complete graph on n vertices.

Theorem 12 *Let $a^Tx\leqslant a_0$ define a facet of Q_T^n satisfying $a\geqslant 0$. Let C be a clique in the graph $G(a)$ induced by the edges $E(a):=\{\{i,j\}\in E \mid a_{ij}>0\}$ with $|C|\geqslant 3$. Suppose that every vertex $v\in C$ is contained in one additional clique C_v of cardinality 2 in $G(a)$, say $C_v=\{v,v'\}$, and assume that for every $v\in C$, $a_{v'i}=0$ for all $i\in V-C_v$ and that $a_{ij}=\alpha$ for every edge $\{i,j\}\in E(a)$ with $\{i,j\}\cap C\neq\emptyset$.*

(a) *Now add four vertices $n+1,\ldots,n+4$ (two new teeth of size 2 each) and define a new inequality in the following way:*

$$\begin{aligned}
&a_{ij}^*:=a_{ij} && \text{for all } \{i,j\}\in E-E(C),\\
&a_{ij}^*:=\alpha && \text{for all } \{i,j\}\in E(C\cup\{n+1,n+2\}),\\
&a_{n+1,n+3}^*:=a_{n+2,n+4}^*=\alpha, &&\\
&a_{ij}^*:=0 && \text{otherwise},\\
&a_0^*:=a_0+3\alpha. &&
\end{aligned}$$

Then $a^{*T}x\leqslant a_0^*$ defines a facet of Q_T^{n+4}.

(b) *Add one vertex $n+1$ (to C) and set $a_0^*:=a_0+\alpha$, $a_{ij}^*:=a_{ij}$ for all $\{i,j\}\in E$, $a_{i,n+1}^*:=\alpha$ for all $i\in C$ and $a_{ij}^*:=0$ otherwise. Then, $a^{*T}x\leqslant a_0^*$ defines a facet of Q_T^{n+1}.*

Theorem 13 *Let $a^Tx\leqslant a_0$ define a facet of Q_T^n satisfying $a\geqslant 0$, and let C be a clique in the graph $G(a)$. Let $Z:=\{v\in C \mid a_{vi}=0$ for all $i\in V-C\}$ and let $Y:=\{v\in V-C \mid \exists w\in C-Z$ such that $a_{vw}=0\}$ (if $C=Z$, then $Y:=V-C$). Suppose one of the following conditions is satisfied:*

(i) *$|Z|\geqslant 2$, $a_{ij}=\alpha$ for all $\{i,j\}\in E$ with $\{i,j\}\cap Z\neq\emptyset$ and $a_{ij}\geqslant\alpha$ for all $\{i,j\}\in E(C-Z)$; or*

(ii) *$|Z|=1$, $|Y|\geqslant 2$ and $a_{ij}=\alpha$ for all $\{i,j\}\in E$ with $\{i,j\}\cap C\neq\emptyset$.*

Set $a_0^*:=a_0+\alpha$, $a_{ij}^*:=a_{ij}$ for all $\{i,j\}\in E$, $a_{i,n+1}^*:=\alpha$ for all $i\in C$ and $a_{ij}^*:=0$ otherwise, then $a^{*T}x\leqslant a_0^*$ defines a facet of Q_T^{n+1}.

By employing some additional lifting theorems of the above type, it was shown by Grötschel & Padberg [1979b] that all comb inequalities define facets of Q_T^n.

Grötschel & Pulleyblank [1985] use this result to show by means of a technically involved double induction that all clique tree inequalities define facets of Q_T^n.

By combining the result of Exercise 15 and the fact that clique tree inequalities define facets of Q_T^n, Theorem 9 implies that all clique tree inequalities also define facets of the monotone polytope $\tilde{Q}_T^n$. Moreover, it was shown by Grötschel & Pulleyblank [1985] that among a well-defined large class of valid inequalities for Q_T^n (resp. $\tilde{Q}_T^n$), the clique tree inequalities are the only ones defining facets of Q_T^n (resp. $\tilde{Q}_T^n$).

Furthermore, except for some obvious cases, no two clique tree inequalities are equivalent with respect to Q_T^n. (In considering the exceptions, see Lemma 8(a) and also compare this to Lemma 3.)

Let us now summarize all the results discussed so far.

Theorem 14 *Let $K_n = (V, E)$ be the complete graph on $n \geq 6$ vertices and let $\mathcal{W}$ be a set of vertex sets in K_n such that for all $W \in \mathcal{W}$, $3 \leq |W| \leq n-3$, and $W \in \mathcal{W}$ if and only if $V - W \notin \mathcal{W}$. Then the following is a system of facet-defining inequalities for Q_T^n, no two of which are equivalent:*

(a) $x_{ij} \geq 0$ *for all* $\{i, j\} \in E$,

(b) $x_{ij} \leq 1$ *for all* $\{i, j\} \in E$,

(c) *subtour elimination constraints:*

$$x(E(W)) \leq |W| - 1 \qquad \text{for all } W \in \mathcal{W},$$

(d) *comb inequalities:*

$$x(E(H)) + \sum_{j=1}^{s} x(E(T_j)) \leq |H| + \sum_{j=1}^{s} (|T_j| - 1) - \frac{s+1}{2}$$

for all $H, T_1, \ldots, T_s \subseteq V$ *satisfying*

(d_1) $|H \cap T_j| \geq 1, \quad j = 1, \ldots, s,$

(d_2) $|T_j - H| \geq 1, \quad j = 1, \ldots, s,$

(d_3) $T_i \cap T_j = \emptyset, \quad 1 \leq i \leq j \leq s,$

(d_4) $s \geq 3$ *and odd,*

(d_5) $H \in \mathcal{W}$,

(e) *clique tree inequalities (with at least two handles):*

$$\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) \leq \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \frac{s+1}{2}$$

for all handles $H_1, \ldots, H_r$, $r \geq 2$, *and teeth* $T_1, \ldots, T_s$ *satisfying the definition in Section 4.2.*

Moreover, the degree equations

$$x(\delta(i)) = 2, \qquad i = 1, \ldots, n$$

form a minimal equation system for Q_T^n.

Theorem 15 *The following is a nonredundant system of facet-defining inequalities for* $\tilde{Q}_T^n$, $n \geq 6$:

(a) $x_{ij} \geq 0$ *for all* $\{i, j\} \in E$,

(b) *degree constraints:*

$$x(\delta(i)) \leq 2 \qquad \textit{for all } i \in V,$$

(c) *subtour elimination constraints:*

$$x(E(W)) \leq |W|-1 \qquad \textit{for all } W \subseteq V,\ 2 \leq |W| \leq n-1,$$

(d) *clique tree inequalities (with at least one handle):*

$$\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) \leq \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \frac{s+1}{2}$$

for all clique trees with at least one handle defined in Section 4.2.

By comparing the inequalities listed in Theorems 14 and 15 with those defining Q^n_{1T}, $\tilde{Q}^n_{1T}$ (cf. (12), . . . , (18)), and Q^n_{2M}, $\tilde{Q}^n_{2M}$ (cf. Theorems 4 and 3), one can easily see which of the facet-defining inequalities for the 1-tree (resp. 2-matching) polytopes define facets of the symmetric traveling salesman polytopes. More precisely (cf. Grötschel [1977a]), the inequalities from (12), . . . , (18) and Theorems 3 and 4 which are missing in Theorem 14 (in Theorem 15, respectively) are redundant with respect to Q^n_T (to $\tilde{Q}^n_T$, respectively). In particular, we can make the following remarks.

Remark 1

(a) *Every nonredundant system of facet-defining subtour elimination constraints* (18) *for* Q^n_{1T} *is a nonredundant system of facet-defining subtour elimination constraints for* Q^n_T.

(b) *Every facet-defining inequality for* $\tilde{Q}^n_{1T}$ *is also facet defining for* $\tilde{Q}^n_T$.

(c) *All facet-defining 2-matching constraints for* $\tilde{Q}^n_{2M}$ *except for those with* $k=1$ *(one tooth) are facet-defining for* $\tilde{Q}^n_T$.

(d) Q^n_T *and* Q^n_{2M} *have the same affine hull and for every nonredundant complete system of inequalities and equations for* Q^n_{2M}, *the corresponding system without the 2-matching inequalities having* $k=1$ *(one tooth) is a nonredundant facet-defining system for* Q^n_T.

4.4 'Bad' facets of Q^n_T and $\tilde{Q}^n_T$

We shall now discuss several classes of inequalities, termed 'bad', which define facets for Q^n_T or $\tilde{Q}^n_T$ and which are thus, by definition, required in any complete linear description of the respective polytopes. We do, however, have reason to believe that they are of little practical use in cutting plane algorithms for the TSP.

These inequalities are usually defined by certain properties which – unless $\mathcal{P} = \mathcal{NP}$ – cannot be checked in polynomial time. Some of these properties are not even known to be in $\mathcal{NP}$ or co-$\mathcal{NP}$. In some (intuitive) sense these inequalities provide a polyhedral explanation for the 'intractability' of the TSP (see Chapter 3 for a precise version of this statement).

Let $\tilde{\mathscr{S}}_n$ be the set of subsets of tours in $K_n = (V, E)$. As mentioned in Section 2, $\tilde{\mathscr{S}}_n$ is an independence system on the edge set E of K_n. With every subset $F \subseteq E$ we can associate a number $r(F)$ called *rank* of F as follows:

$$r(F) := \max\{|S| : S \subseteq F \text{ and } S \in \tilde{\mathscr{S}}_n\}.$$

A moment's thought shows that for every subset $F \subseteq E$ the inequality

$$x(F) \leq r(F)$$

is valid and supporting with respect to $\tilde{Q}^n_{\mathrm{T}}$. This way of defining valid and supporting 0–1 inequalities is applicable to any other combinatorial optimization problem in an analogous manner. The inequalities of the type $x(F) \leq r(F)$ are usually called *rank inequalities*. As a matter of fact, all inequalities for the TSP encountered so far are rank inequalities except for those clique tree inequalities which contain a tooth and a handle meeting in more than one vertex.

It is of course not apparent how to compute the rank of a set F; in fact this is as hard as the TSP itself, and assuming the rank is known, it is not easy to decide whether or not a rank inequality defines a facet of the TSP. There are two obvious necessary conditions (valid for general independence systems and not only for the TSP) which we would like to mention.

A set $F \subseteq E$ is *closed* if $r(F) < r(G)$ for all G which strictly contain F, and F is called *inseparable* if there are no two disjoint nonempty subsets F_1, F_2 of F with $F = F_1 \cup F_2$ and $r(F) = r(F_1) + r(F_2)$.

Lemma 9 *If the inequality* $x(F) \leq r(F)$ *defines a facet of* $\tilde{Q}^n_{\mathrm{T}}$, *then* F *is closed and inseparable.*

We next study some graphs whose edge sets give rise to rank inequalities defining facets of $\tilde{Q}^n_{\mathrm{T}}$. If v is a vertex of a graph G, then $G - v$ denotes the graph obtained by removing v.

Let $G = (W, F)$ be a graph.

(a) G is called *Hamiltonian* if it contains a Hamiltonian cycle.
(b) G is called *semi-Hamiltonian* if it contains a Hamiltonian path.
(c) G is called *maximal non-Hamiltonian* (*maximal non-semi-Hamiltonian*) if G is non-Hamiltonian (not semi-Hamiltonian) but if the addition of any edge (not in G) to G makes the new graph Hamiltonian (semi-Hamiltonian).
(d) G is called *hypo-Hamiltonian* (*hypo-semi-Hamiltonian*) if G is non-Hamiltonian (non-semi-Hamiltonian) and $G - v$ is Hamiltonian (semi-Hamiltonian) for all $v \in W$.
(e) G is called *maximal hypo-Hamiltonian* (*maximal hypo-semi-Hamiltonian*) if G is hypo-Hamiltonian (hypo-semi-Hamiltonian) and maximal non-Hamiltonian (maximal non-semi-Hamiltonian).

For the history of hypo-Hamiltonian and hypo-semi-Hamiltonian graphs

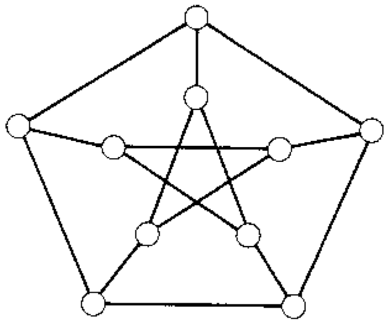


Figure 8.4

see Thomassen [1974, 1978] and Grötschel [1977a]. To mention some results, it is known that the smallest hypo-Hamiltonian graph has 10 vertices and is unique. This graph is the famous Petersen graph which is given in Figure 8.4. There are hypo-Hamiltonian graphs with n vertices for $n = 10, 13, \ldots, 16, 18$ and larger. There are no hypo-Hamiltonian graphs with 11 or 12 vertices and no such graph with 9 or fewer vertices. The only open case is $n = 17$.

Hypo-semi-Hamiltonian graphs of order n are known for $n = 34$, 37 and for each $n \geq 39$. One can show that there are no hypo-semi-Hamiltonian graphs with fewer than 12 vertices. All other cases are open.

It is obvious that for any non-Hamiltonian graph $G = (W, F)$ with k vertices, the inequality

$$x(F) \leq |W| - 1 \tag{35}$$

is valid for $\tilde{Q}_T^n$, $n \geq k$, and that in case G is maximal non-Hamiltonian the inequality is supporting. However, if $n > k$, then the subtour elimination constraint for W implies (35), i.e. $x(F) \leq x(E(W)) \leq |W| - 1$. So, (35) is a candidate for a facet only in the case $k = n$. Clearly, if $G = (W, F)$ is maximal non-Hamiltonian then F is closed, but F may be separable. (Consider the graph K_{n-1} plus an edge, say $\{n-1, n\}$.)

Research problem *Characterize those maximal non-Hamiltonian graphs of order n for which* (35) *defines a facet of $\tilde{Q}_T^n$.*

By definition, all maximal hypo-Hamiltonian graphs are maximal non-Hamiltonian graphs. It is not even known whether all maximal hypo-Hamiltonian graphs define facets of $\tilde{Q}_T^n$, but there is a useful sufficient condition known.

Definition Let $G = (W, F)$ be a hypo-Hamiltonian (or hypo-semi-Hamiltonian) graph. A vertex $v \in W$ is said to have *property* Δ if for any two neighbors v_1, v_2 of v, one of the following conditions is satisfied:

(a) $G - v_1$ contains a Hamiltonian cycle (path) containing edge $\{v, v_2\}$.
(b) $G - v_2$ contains a Hamiltonian cycle (path) containing edge $\{v, v_1\}$.

(c) There exists a neighbor v_3 of v such that both $G-v_1$ and $G-v_2$ contain a Hamiltonian cycle (path) containing edge $\{v, v_3\}$.

G has *property* Δ if every vertex in W has property Δ.

It was shown by Grötschel [1980a] that almost all known hypo-Hamiltonian and hypo-semi-Hamiltonian graphs have property Δ. Moreover, the following result was proved.

Theorem 16 *Let $G=(W,F)$ be a hypo-Hamiltonian graph of order n having property Δ. Let $G'=(W,F')$ be any maximal hypo-Hamiltonian graph with $F\subseteq F'$, then*

$$x(F')\leqslant n-1$$

defines a facet of $\tilde{Q}_{\mathrm{T}}^{n}$ (but not a facet of $\tilde{Q}_{\mathrm{T}}^{k}$, $k>n$).

Note that the hypo-Hamiltonian graphs of order n in Theorem 16 induce a kind of inequality which are peculiar to $\tilde{Q}_{\mathrm{T}}^{n}$. All inequalities of the type $x(F)\leqslant r(F)$ encountered so far have the property that if they define a facet of $\tilde{Q}_{\mathrm{T}}^{n}$ they also define a facet of $\tilde{Q}_{\mathrm{T}}^{k}$, $k\geqslant n$. This is not the case for hypo-Hamiltonian inequalities.

To give an example, the Petersen graph is known to be a maximal hypo-Hamiltonian graph having property Δ. Thus, if F is the edge set of a Petersen graph in K_{10}, then $x(F)\leqslant 9$ defines a facet of $\tilde{Q}_{\mathrm{T}}^{10}$. Maurras [1976] showed that this Petersen inequality also defines a facet of Q_{T}^{10}. Moreover, he proved that if the two vertices of an edge in the Petersen graph are replaced by a clique of size $k\geqslant 2$, then the corresponding inequality $x(F)\leqslant n+k-3$ defines a facet of Q_{T}^{n+k-2}, $n\geqslant 10$. This is the only known case of such 'bad' inequalities which also define facets of Q_{T}^{n}.

Research problem *Which inequalities induced by maximal non-Hamiltonian graphs define facets of Q_{T}^{n}?*

Statements similar to the ones made about maximal non-Hamiltonian graphs can be made with respect to hypo-semi-Hamiltonian and maximal non-semi-Hamiltonian graphs. The main result of Grötschel [1980a] about this type of graphs is the following.

Theorem 17 *Let $G=(W,F)$ be a hypo-semi-Hamiltonian graph of order n having property Δ, and let $G'=(W,F')$ be a maximal hypo-semi-Hamiltonian graph with $F\subseteq F'$; then*

$$x(F')\leqslant n-2$$

defines a facet of $\tilde{Q}_{\mathrm{T}}^{k}$ for all $k\geqslant n$.

It should be noted that the number of nonisomorphic hypo-Hamiltonian (hypo-semi-Hamiltonian, respectively) graphs with property Δ is not at all small, i.e. that every possible maximal graph containing it and labeled to give a different subgraph of K_n defines a different facet of $\tilde{Q}_{\mathrm{T}}^{n}$. Thus the number of such facet-defining inequalities is large in general.

On the other hand, for some small cases we have computed the number of tours whose incidence vectors satisfy the hypo-Hamiltonian (hypo-semi-Hamiltonian, respectively) inequalities with equality, and it turned out that these numbers are rather small compared to the number of tours whose incidence vectors satisfy the subtour elimination constraints, say, with equality. This is a more or less intuitive explanation of the fact that such inequalities were not needed to prove optimality in cutting plane procedures, cf. Chapter 9.

Papadimitriou & Yannakakis [1984] define a further class of bad facets. Call a vertex v of a graph a *supernode* if v is adjacent to every other vertex. Let $G=(W,F)$ be a graph of order n without a supernode. Construct a graph $G'=(W',F')$ with $3n$ vertices as follows. Replace every vertex $v \in W$ by three mutually adjacent vertices v_1, v_2, v_3. G' has the following edges: all edges $\{u_3, v_3\}$, two edges $\{u_3, v_1\}$, $\{v_3, u_1\}$ if $\{u, v\} \in F$ and an edge $\{u_3, v_2\}$ if all neighbors of v in G are also neighbors of u. Papadimitriou & Yannakakis [1984] prove the following results.

Theorem 18 *Let $G=(W,F)$ be a graph of order n without a supernode. If G is maximal non-Hamiltonian then $G'=(W',F')$ is maximal non-Hamiltonian. Moreover*

$$x(F') \leq 3n-1$$

is a facet of $\tilde{Q}_T^{3n}$ if and only if G is maximal non-Hamiltonian.

So Theorem 18 shows how to modify a maximal non-Hamiltonian graph to make the corresponding inequality facet-inducing. In addition to Theorems 16 and 17, Theorem 18 gives a further class of bad facets of $\tilde{Q}_T^n$.

It is easy to see that none of the inequalities introduced in this section is equivalent to any of the inequalities described in Theorems 14 and 15. By adding all the facet-inducing hypo-Hamiltonian, hypo-semi-Hamiltonian inequalities, etc., to the systems in Theorems 14 and 15, we obtain better linear descriptions of Q_T^n (of $\tilde{Q}_T^n$, respectively) even though we have no claim (nor conjecture) as to the completeness of the linear system thus obtained.

4.5 Further remarks

We have already mentioned some open problems about Q_T^n and $\tilde{Q}_T^n$, which might be solvable with some effort. There are many more interesting questions which one could ask but most of them seem to be hopelessly difficult. Let us mention one of those.

Research problem *Characterize all (or some interesting) 0–1 inequalities which define facets of Q_T^n (resp. $\tilde{Q}_T^n$).*

The 0–1 inequalities are simply the rank inequalities with respect to the independence system $\tilde{\mathcal{T}}_n$ (cf. the beginning of Section 4.4).

The usual way to exhibit new classes of facet-inducing inequalities is to consider small examples, i.e. Q_T^n for small n, and to try to find a complete inequality system. If the system one has at hand is not complete, then this system must have some fractional solutions. By investigating the fractional solutions one may be able to obtain new valid inequalities which cut off fractional solutions and which might be the basic examples of a new class of interesting inequalities.

Let us therefore mention the values of n for which complete systems for Q_T^n and $\tilde{Q}_T^n$ are known, and compare these results to Theorems 14 and 15.

For $n = 3$, 4 and 5, it is trivial to see that $Q_T^n = Q_{2M}^n$, and hence complete and nonredundant systems for Q_T^n, $n = 3, 4, 5$, are known from Theorem 4 and Exercise 7.

The case $n = 6$ is the first where $Q_T^n \neq Q_{2M}^n$ and where subtour elimination constraints have to be used. It is not too hard to prove that the inequality system given in Theorem 14, i.e. trivial constraints, 2-matching inequalities and subtour elimination constraints, is complete and nonredundant for Q_T^6.

However, the three types of inequalities are not sufficient for Q_T^7. Here comb inequalities with teeth of size 3 have to be used. This follows from a result of Norman [1955] which is discussed in detail by Grötschel [1977a, pp. 144–145]. But still, the system of Theorem 14 is complete and nonredundant for Q_T^7.

No further completeness results about Q_T^n are known. We believe that the system of Theorem 14 is complete and nonredundant for Q_T^8 and Q_T^9, but we have no proof.

For $n = 10$, the Petersen graph inequalities (cf. Section 4.4) define facets, so the system of Theorem 14 is not complete for Q_T^{10}.

The first time where clique tree inequalities, which are not comb inequalities, enter is the case $n = 11$. To give an example, consider the polytope $P \subseteq \mathbb{R}^{55}$ ($n = 11$) defined by the degree equations (5), the trivial inequalities in Theorem 8, the subtour elimination constraints (10) and the 2-matching inequalities (25); then P has the fractional vertex shown in Figure 8.5(a). If $P' \subseteq \mathbb{R}^{55}$ is the polytope obtained from P by adding the comb inequalities (34) then P' has the fractional vertex shown in Figure 8.5(b). We leave it as an exercise to the reader to find the comb and clique tree inequalities that are violated by the vertices depicted in Figures 8.5(a) and (b), respectively.

Another way to look at Q_T^n is from the subtour elimination point of view. Namely, for $n = 3, 4, 5$, Q_T^n is given by the degree equations and trivial inequalities (2-matching constraints are not needed, and for $n = 5$, subtour elimination constraints for $|W| = 3$ are equivalent to trivial inequalities.) So one may ask up to which n do the subtour elimination constraints of Theorem 11 (and trivial inequalities and degree equations) determine Q_T^n. In fact, subtour elimination constraints are not sufficient for Q_T^6. This has already been observed by Held & Karp [1970], but no polyhedral interpretation was given there. Held & Karp [1970] have shown that a Lagrangean relaxation method finds the optimum value of any minimization problem over the 1-tree polytope Q_{1T}^n intersected with the affine space defined by the degree equations (5). (Recall the complete description of Q_{1T}^n and Q_T^n given

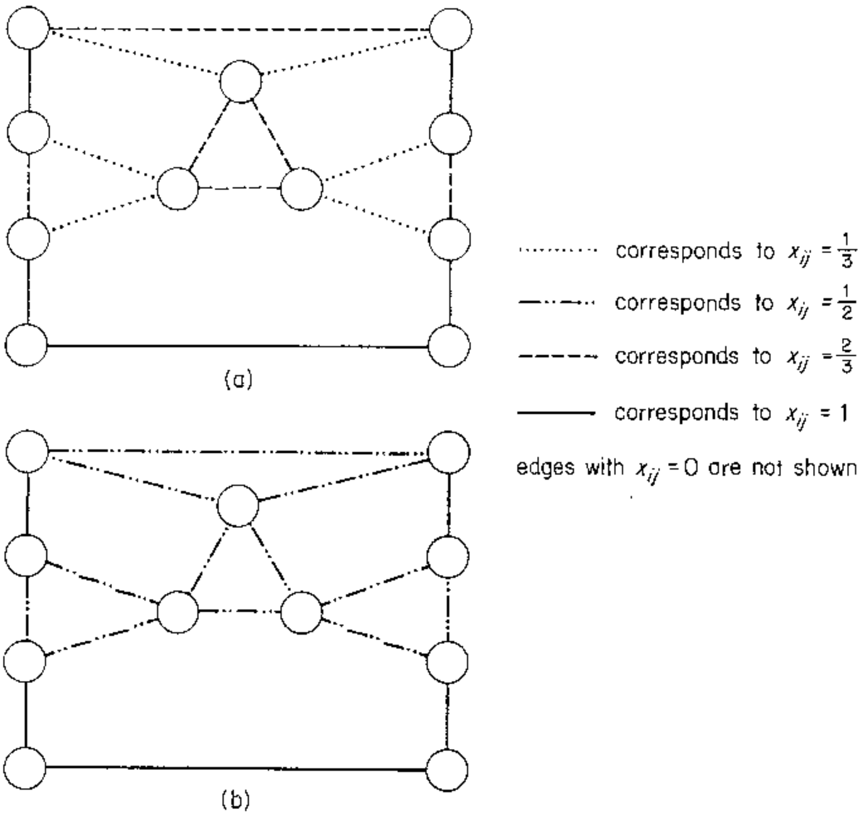


Figure 8.5

in (15), (16), (17) and also recall the close relationship between Q^n_{1T} and Q^n_T as was observed in Remark 1(a).) Held and Karp observed that if the objective function shown in Figure 8.6 is minimized, the optimal tour length is 4 while the relaxed problem mentioned above has an optimum value 3. (We are in the case $n = 6$.)

The edges in Figure 8.6 have the weight assigned to them as shown; all other edges of K_6 have a large positive weight. In fact, we may assume that all other edges have weight 2. Let us denote the $\{0, 1, 2\}$-valued objective function defined this way by $c^T x$.

We claim that $-c^T x \leqslant -4$ defines a facet of Q^6_T which is equivalent to a 2-matching constraint. This can be seen as follows.

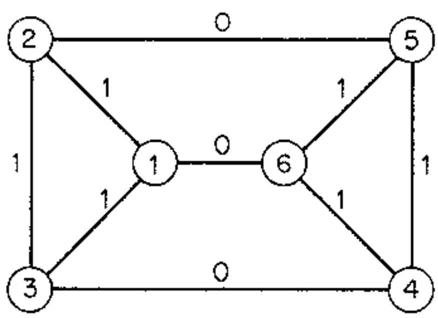


Figure 8.6

Denote by $a^T x \leq a_0 = 4$ (resp. $b^T x \leq b_0 = 4$) the two facet-defining 2-matching constraints determined by the handle $H = \{1, 2, 3\}$ and the teeth $T_1 = \{1, 6\}$, $T_2 = \{2, 5\}$, $T_3 = \{3, 4\}$, by $H' = \{4, 5, 6\}$ and $T_1' = T_1$, $T_2' = T_2$, $T_3' = T_3$, respectively. Then we get

$$-c^T x = a^T x + b^T x - \sum_{i=1}^{6} x(\delta(i)),$$

and

$$-4 = a_0 + b_0 - 12.$$

By Lemma 3, $a^T x \leq 4$ and $b^T x \leq 4$ are equivalent with respect to Q_{2M}^n, and since $\text{aff}(Q_{2M}^n) = \text{aff}(Q_T^n)$ these inequalities are also equivalent with respect to Q_T^n. Thus $c^T x \geq 4$ is a further equivalent version of the two 2-matching constraints with respect to Q_T^6.

So from the polyhedral point of view it is clear that the objective function shown in Figure 8.6 leads to a fractional solution of the relaxed problem, since an objective function is minimized which induces a facet of Q_T^n but no equivalent version of the facet-defining inequality is contained in the system defining the polytope corresponding to the relaxed problem.

With respect to $\tilde{Q}_T^n$ almost no investigations have been made about the completeness of the system of inequalities given in Theorem 15. It is trivial to see that $\tilde{Q}_T^3 = \tilde{Q}_{2M}^3$ is the unit hypercube in $\mathbb{R}^3$. In case $n = 4$, we do already have $\tilde{Q}_T^4 \neq \tilde{Q}_{2M}^4$ since the incidence vectors of 3-cycles are in $\tilde{Q}_{2M}^4$ but not in $\tilde{Q}_T^4$. It is shown by Grötschel [1977b] that $\tilde{Q}_T^4$ is given by the trivial inequalities, the degree inequalities and the four subtour elimination constraints (on vertex sets of cardinality 3), i.e. the system in Theorem 15 is complete and nonredundant for $n = 4$. We state our conjecture about 'small $\tilde{Q}_T^n$' in the following problem.

Research problem *Prove that the system of linear inequalities in Theorem 15 is complete (it is known to be nonredundant!) for $\tilde{Q}_T^n$ for $n = 5, \ldots, 9$.*

In this chapter we have discussed only the TSP defined on the complete graph K_n. One may as well study the polytope $Q_T(G) := \text{conv}\{x^T \in \mathbb{R}^E \mid T$ is a Hamiltonian cycle in $G\}$ where $G = (V, E)$ is an arbitrary graph. (Similarly, $\tilde{Q}_T(G)$ can be defined.) Since the Hamiltonian graph problem is $\mathcal{NP}$-complete, it is $\mathcal{NP}$-complete to decide whether $Q_T(G)$ is nonempty. Therefore it seems quite hard to say anything reasonable about the facet structure of $Q_T(G)$.

However, for special classes of graphs, complete inequality systems for $Q_T(G)$ might be easy to describe. For instance, Barahona & Grötschel were able to characterize $Q_T(G)$ completely for all graphs G not contractible to the complete graph K_6 minus an edge. Cornuéjols, Naddef & Pulleyblank [1983] have recently studied the class of graphs G such that $Q_T(G)$ is given by the trivial inequalities, the degree equations, and one equation for each 3-edge cut-set. This class contains $K_{3,3}$, wheels, Halin graphs and some other graphs. More generally, we may ask the following question.

Research problem *What is the class of graphs G such that $Q_T(G)$ is determined by the system described in Theorem 14 or a subsystem thereof?*

Cornuéjols, Naddef & Pulleyblank [1983] moreover proved the following very interesting 3-cut set composition theorem for traveling salesman polytopes.

Theorem 19 *Let $G_1=(V_1,E_1)$, resp. $G_2=(V_2,E_2)$ be graphs each having a vertex of degree 3; say $i\in V_1$ lies on the edges $\{i,i_1\}$, $\{i,i_2\}$, $\{i,i_3\}\in E_1$ and $j\in V_2$ on the edges $\{j,j_1\}$, $\{j,j_2\}$, $\{j,j_3\}\in E_2$. Let $G=G_1*G_2=(V,E)$ be the graph defined as follows: $V=(V_1\cup V_2)-\{i,j\}$, $E=((E_1\cup E_2)-\{\{i,i_k\},\{j,j_k\}\mid k=1,2,3\})\cup\{\{i_1,j_1\},\{i_2,j_2\},\{i_3,j_3\}\}$. Then a complete system of equations and inequalities for $Q_T(G)$ is obtained by juxtaposing the inequalities and equations which define $Q_T(G_1)$ and $Q_T(G_2)$ and identifying $x_{ii_1}=x_{jj_1}=x_{i_1j_1}$, $x_{ii_2}=x_{jj_2}=x_{i_2j_2}$,* and *$x_{ii_3}=x_{jj_3}=x_{i_3j_3}$.*

Such investigations may lead to new classes of graphs for which the TSP is solvable in polynomial time. For instance, we shall show in Chapter 9 that we can optimize in polynomial time over the trivial inequalities, the subtour elimination constraints, the 2-matching constraints and the degree equations. Thus, the TSP is solvable in polynomial time for all graphs G for which $Q_T(G)$ is completely determined by these equations and inequalities. But which are these graphs? More modestly put, the problem is to find (large) classes of graphs for which $Q_T(G)$ can be described this way!

Another way to use the facet-inducing inequalities is the following. Clearly, if $G=(V,E)$ has n vertices, then every inequality valid for Q_T^n is also valid for $Q_T(G)$ (the variables corresponding to edges in K_n which are not in G have to be deleted, of course). Using Farkas' lemma (or equivalent theorems of the alternative) it is sometimes easy to show that a system of inequalities valid for $Q_T(G)$ (e.g. all known ones for Q_T^n) has no solution. This in turn implies that $Q_T(G)$ is empty and hence that G is not Hamiltonian. (This proof technique is for instance described by Chvátal [1973a]). We think that these remarks are quite important, since there are only few other (and not very powerful) methods known to show that a given graph is non-Hamiltonian. (For a sampling of other techniques, see Chapter 11.)

Exercises

11. (a) Prove that none of the inequalities $x_{ij}\geqslant 0$ defines a facet for Q_T^3 and Q_T^4.
(b) Prove that no two inequalities $x_{ij}\leqslant 1$ and $x_{pq}\leqslant 1$, $\{i,j\}\neq\{p,q\}$, are equivalent for Q_T^n, $n\geqslant 5$, but that when $n=4$, $x_{ij}\leqslant 1$ and $x_{pq}\leqslant 1$ are equivalent if all vertices i,j,p,q are different.
(c) Prove that $Q_T^n=Q_{2M}^n$ for $n=3,4,5$.
12. Prove Lemma 7.
13. Prove Lemma 8.

14. (a) Show that the 2-matching inequalities (22) with $k=1$ are implied by the subtour elimination constraints.
(b) Use the fact that the smallest comb (cf. Figure 8.3) defines a facet of Q_T^n, $n \geqslant 6$, and Theorems 12 and 13 to show that all 2-matching inequalities (22) with $k \geqslant 3$ define facets of Q_T^n. (*Hint*: The handle of the comb corresponding to a 2-matching constraint should be used as the clique C in Theorems 12 and 13; see Grötschel & Padberg [1979a, 1979b].)
15. Prove that all clique tree inequalities (33) are support reduced (cf. the procedure preceding Lemma 6).
16. Prove Lemma 9.
17. (a) Find a comb inequality which is violated by the point $x \in \mathbb{R}^{55}$ shown in Figure 8.5(a).
(b) Find a clique tree inequality which is violated by the point $x \in \mathbb{R}^{55}$ shown in Figure 8.5(b).
18. Let $G=(V, E)$ be the graph consisting of the vertices $V=\{1, 2, \ldots, 8\}$ and the edges $E=\{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{1,5\}, \{5,6\}, \{6,7\}, \{7,8\}, \{5,8\}, \{4,8\}\}$. Clearly, $Q_T(G)$ is contained in the polyhedron Q given by

$$x(\delta(i))=2, \quad i=1,\ldots,8,$$

$$\left.\begin{aligned} x(E\{1,2,3,4\}) &\leqslant 3 \\ x(E\{5,6,7,8\}) &\leqslant 3 \end{aligned}\right\} \quad (\textit{two subtour elimination constraints}),$$

$$x_{ij} \leqslant 1 \quad \text{for all } \{i,j\} \in E,$$

$$x_{ij} \geqslant 0 \quad \text{for all } \{i,j\} \in E.$$

Prove that $Q_T(G)$ is empty (i.e. G is not Hamiltonian) by showing that Q is empty. (*Hint*: Use Farkas' Lemma and exhibit a solution of the system dual to the system defining Q. Recall that Farkas' lemma states that either the primal system $Dx=d$, $Ax \leqslant b$, $x \geqslant 0$ or the dual system $u^T D + v^T A \geqslant 0$, $v \geqslant 0$, $u^T d + v^T b < 0$ has a solution, but never both.)

5 THE ASYMMETRIC TRAVELING SALESMAN POLYTOPES

The asymmetric traveling salesman polytopes P_T^n and $\tilde{P}_T^n$ have not received as much attention in the literature as the corresponding symmetric ones. Perhaps as a result, most computational studies of the asymmetric TSP utilize no more polyhedral information than is provided by the subtour elimination constraints. While asymmetric TSPs appear to be easier to solve than their symmetric counterparts of equal size, it is to be expected that the exploitation of other classes of facet-defining inequalities as well as of several other interesting properties of P_T^n and $\tilde{P}_T^n$ known to date, should push the problem-solving capabilities for the TSP beyond its current limits.

The results reported in this section are mainly due to Grötschel [1977a], Grötschel & Padberg [1974, 1975b, 1977], Grötschel & Wakabayashi [1981a, 1981b] and Padberg & Rao [1974]. In the following, $D_n=(V, A)$ denotes the complete digraph on n vertices.

5.1 Basic properties of P_T^n and $\tilde{P}_T^n$, sequential lifting

The asymmetric traveling salesman polytope P_T^n is contained in the assignment polytope P_A^n (cf. Theorem 5), and we know from Proposition 3 that $\dim(P_A^n) = |A| - 2|V| + 1$. Thus we have an upper bound for the dimension of P_T^n. In fact we have the following theorem.

Theorem 20 $\dim(P_T^n) = |A| - 2|V| + 1 = (n-1)^2 - n$, *for* $n \geqslant 3$.

The result above has been stated in two abstracts [Heller, 1953; Kuhn, 1955a]. A direct proof of Theorem 20 analogous to the first proof of Theorem 7 has been given by Grötschel & Padberg [1977]. It is not too difficult to give a proof of Theorem 20 paralleling the second proof of Theorem 7. From the $|E| - |V| + 1$ undirected tours constructed in the first proof of Theorem 7 we can obtain $|A| - 2|V| + 2$ directed tours by taking the two possible orientations of each undirected tour. In order to complete this approach we must show that the incidence vectors of these directed tours are linearly independent, and this is left as an exercise.

Since the subtour elimination constraints for $|W| = 2$, i.e. $x_{ij} + x_{ji} \leqslant 1$, are valid with respect to P_T^n (cf. (11)), it is obvious that the upper bounds $x_{ij} \leqslant 1$ do not define facets of P_T^n. However, the nonnegativity constraints do.

Proposition 4 *Let* $n \geqslant 5$, *then* $x_{ij} \geqslant 0$ *defines a facet of* P_T^n *for all* $(i, j) \in A$.

In Section 4.1 we discussed the relations between the symmetric traveling salesman polytopes Q_T^n and $\tilde{Q}_T^n$ and in particular stated Theorem 9, which shows how a facet-defining inequality for $\tilde{Q}_T^n$ can be derived from a facet-defining inequality for Q_T^n. S. Boyd [1984] has generalized Theorem 9 to polyhedra of independence systems arising from monotonization. And thus, by making a valid inequality for P_T^n nonnegative (by adding appropriate multiples of the equation system (7) of Section 2.4) and reducing its support, one can obtain facet-inducing inequalities for $\tilde{P}_T^n$ from facet-inducing inequalities for P_T^n, just as in the symmetric case.

We shall now introduce a technique, called *sequential lifting*, which leads to new facet-defining inequalities for $\tilde{P}_T^n$ and P_T^n. This technique is also applicable to the symmetric TSP but does not produce anything interesting there. The sequential lifting method will be described in a general framework.

Let $\mathscr{I}$ be an independence system (or monotone set system) on a set E, (cf. Section 2.3), and let $P_{\mathscr{I}}$ be the polytope associated with $\mathscr{I}$ (cf. (1)). For every subset $F \subseteq E$, $P_{\mathscr{I}}(F)$ denotes the polytope $\{x \in P_{\mathscr{I}} \mid x_e = 0 \text{ for all } e \in F\}$.

Theorem 21 (Sequential lifting theorem) *Let* $\mathscr{I}$ *be an independence system on* E, *let* $F \subseteq E$ *and let* $e \in F$. *Suppose* $\sum_{k \notin F} a_k x_k \leqslant a_0$ *defines a facet of* $P_{\mathscr{I}}(F)$ *with* $a_0 > 0$. *Set*

$$a_e := a_0 - \max\left\{ \sum_{k \notin F} a_k x_k^I \mid I \subseteq E - F, \{e\} \cup I \in \mathscr{I} \right\}.$$

Then $a_e x_e + \sum_{k \notin F} a_k x_k \leqslant a_0$ *defines a facet of* $P_{\mathscr{I}}(F - \{e\})$.

Theorem 21 was proved by Padberg [1975], who has found various generalizations which we do not discuss here.

The interesting feature of Theorem 21 is the following. Sometimes it is easy to find a set F and an inequality $\sum_{k \notin F} a_k x_k \leq a_0$ such that this inequality defines a facet of $P_{\mathscr{I}}(F)$. Now Theorem 21 tells us what the missing coefficients of a (i.e. the coefficients a_e for all $e \in F$) have to look like such that the extended inequality $\sum_{e \in E} a_e x_e \leq a_0$ defines a facet of $P_{\mathscr{I}}$. Since the coefficient calculation is done sequentially, we may end up with different facets of $P_{\mathscr{I}}$ depending on the order in which the coefficients are lifted. However, it should be observed that the calculation of the new coefficients is a hard problem in general. Only in rare cases the lifted coefficients can be obtained easily.

Let us go back to the asymmetric TSP and let us consider the independence system $\tilde{\mathscr{T}}_n$ of subsets of tours contained in the complete digraph $D_n = (V, A), n \geq 3$.

Proposition 5 *Let C be the arc set of a directed cycle in D_n, $n \geq 3$, of length $k \leq n-1$, and let $F := A - C$. Then the cycle inequality $x(C) \leq |C| - 1$ defines a facet of $\tilde{P}^n_T(F)$.*

Proof Let $C_1, \ldots, C_k$ be the paths of length $k-1$ obtained from C by deleting one arc. Then the incidence vectors of these paths are contained in $\tilde{P}^n_T(F)$ and satisfy $x(C) \leq |C| - 1$ with equality. Let M be the (k, k)-matrix whose columns correspond to the arcs of C and whose rows correspond to the incidence vectors x^{C_i}, $i = 1, \ldots, k$. By permuting rows and columns, M can be transformed into the matrix $E_k - I_k$, where E_k is the (k, k)-matrix whose components are all 1, and where I_k is the (k, k)-identity matrix. $E_k - I_k$ is obviously nonsingular, and thus the k incidence vectors are linearly independent, which proves that $x(C) \leq |C| - 1$ defines a facet of $\tilde{P}^n_T(F)$. □

Now Theorem 21 tells us that every such facet of $\tilde{P}^n_T(F)$ can be lifted to a facet of $\tilde{P}^n_T$. In the analogous case for the symmetric TSP, the inequalities obtained from cycle inequalities by lifting are the subtour elimination constraints. This is not so in the asymmetric case.

It is clear that at every stage of the sequential lifting procedure, the lifting coefficients are 0 for arcs which are not diagonals of C. In Exercise 21, the reader is asked to show that the first time we lift a diagonal arc of a cycle we obtain a coefficient 2. Afterwards we may get a 2, 1 or 0 depending on the order of lifting. But note that in any case we obtain a facet of $\tilde{P}^n_T$ by sequential lifting.

Corollary 4 *All inequalities obtained by lifting cycle inequalities $x(C) \leq |C| - 1$, $2 \leq |C| \leq n-1$, sequentially in any order define facets of $\tilde{P}^n_T$.*

The sequential lifting technique with respect to $\tilde{P}^n_T$ is discussed in detail by Grötschel [1977a]. There is no formula known which describes all the

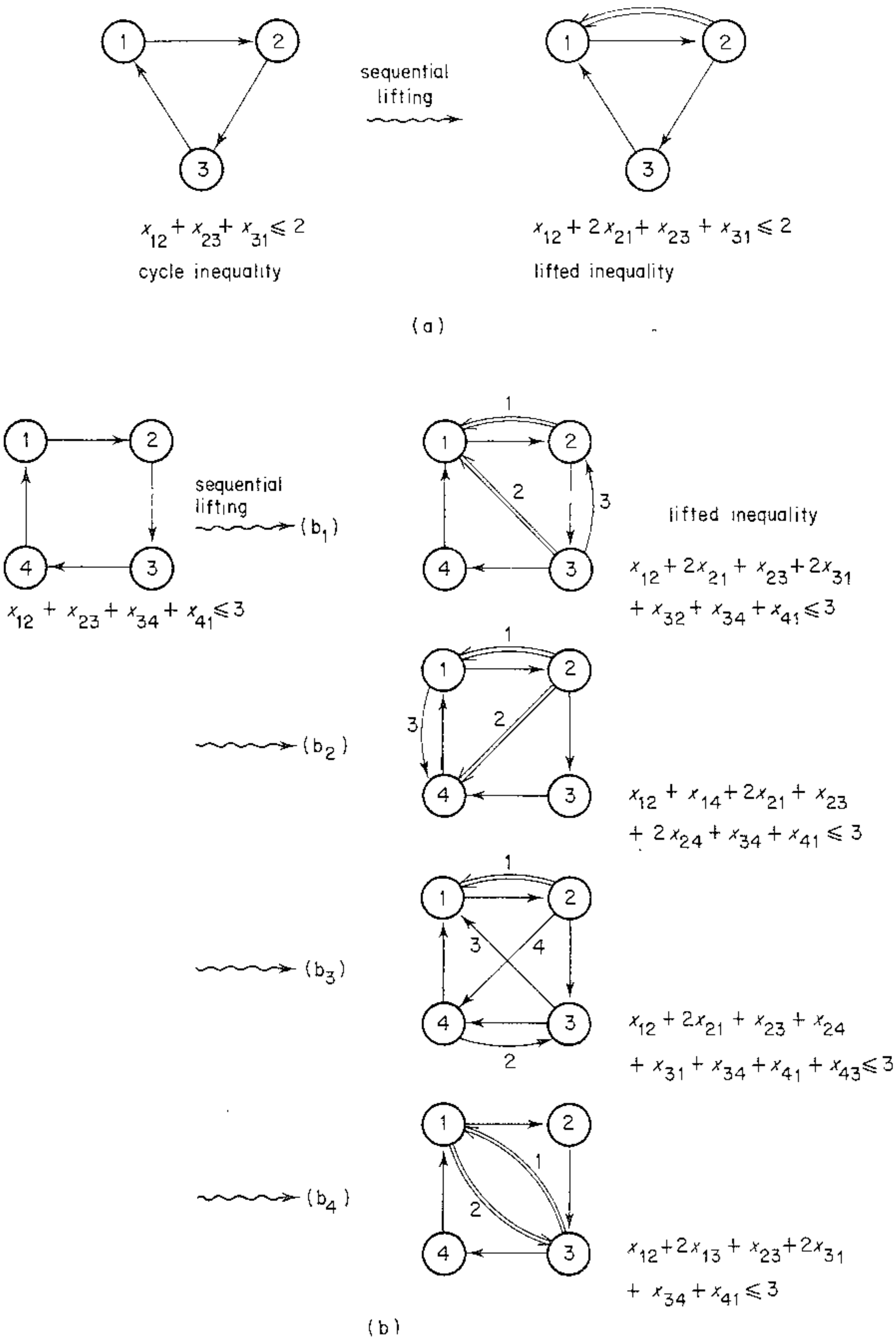

sequential lifting
$x_{12} + x_{23} + x_{31} \leqslant 2$
cycle inequality
$x_{12} + 2x_{21} + x_{23} + x_{31} \leqslant 2$
lifted inequality
(a)
sequential lifting
(b$_1$)
(b$_2$)
(b$_3$)
(b$_4$)
$x_{12} + x_{23} + x_{34} + x_{41} \leqslant 3$
lifted inequality
$x_{12} + 2x_{21} + x_{23} + 2x_{31} + x_{32} + x_{34} + x_{41} \leqslant 3$
$x_{12} + x_{14} + 2x_{21} + x_{23} + 2x_{24} + x_{34} + x_{41} \leqslant 3$
$x_{12} + 2x_{21} + x_{23} + x_{24} + x_{31} + x_{34} + x_{41} + x_{43} \leqslant 3$
$x_{12} + 2x_{13} + x_{23} + 2x_{31} + x_{34} + x_{41} \leqslant 3$
(b)

Figure 8.7

inequalities that can be obtained by this procedure. Figure 8.7 gives some examples.

The arcs in Figure 8.7 drawn as a double line receive a lifting coefficient 2. The numbers on the arcs indicate the order in which the arcs were lifted. Up to isomorphism, the five inequalities defining facets of $\tilde{P}^n_T$, $n \geqslant 3$, resp. $n \geqslant 4$, shown in Figure 8.7 are the only ones that can be obtained from 3-cycles and 4-cycles by sequential lifting. It should be clear that the number of different inequalities obtained from lifting a k-cycle inequality grows quite fast with k.

Grötschel [1977a] and Grötschel & Padberg [1977] explicitly determined various general classes of inequalities valid for $\tilde{P}^n_T$ which can be obtained by lifting k-cycles; we mention two of these.

Theorem 22 *Let* $\{i_1, i_2, \ldots, i_k\} \subseteq V$, $3 \leqslant k \leqslant n-1$, *then*

$$\sum_{j=1}^{k-1} x_{i_j i_{j+1}} + x_{i_k i_1} + 2\sum_{j=2}^{k-1} x_{i_j i_1} + \sum_{j=3}^{k-1} \sum_{h=2}^{j-1} x_{i_j i_h} \leqslant k-1$$

is called a $\overleftarrow{D}_k$*-inequality and*

$$\sum_{j=1}^{k-1} x_{i_j i_{j+1}} + x_{i_k i_1} + 2\sum_{j=3}^{k} x_{i_1 i_j} + \sum_{j=4}^{k} \sum_{h=3}^{j-1} x_{i_j i_h} \leqslant k-1$$

is called a $\overrightarrow{D}_k$*-inequality. All* $\overleftarrow{D}_k$*- and* $\overrightarrow{D}_k$*-inequalities are valid with respect to* $\tilde{P}^n_T$ *and* P^n_T.

5.2 Relations to the symmetric TSP, new valid inequalities

The symmetric TSP is of course a special case of the asymmetric TSP. We shall now study the relations between the symmetric and asymmetric traveling salesman polytopes. Q^n_T and $\tilde{Q}^n_T$ are not subpolytopes or faces of P^n_T and $\tilde{P}^n_T$, respectively, but they are certain projections of the latter polytopes. If we define the mapping $f: \mathbb{R}^A \to \mathbb{R}^E$ (A is the arc set of the complete digraph D_n and E the edge set of K_n) as follows:

$$f(x) = y, \qquad \text{where} \qquad y_{ij} = x_{ij} + x_{ji} \qquad \text{for all } i \neq j,$$

then we have

$$f(P^n_T) = Q^n_T, \qquad \text{and} \qquad f(\tilde{P}^n_T) = \tilde{Q}^n_T.$$

(Note that the order of the indices ij is important for the (directed) variables x_{ij} but not for the (undirected) variables y_{ij}.)

It is easy to see how valid inequalities can be transformed.

Remark 2 *Let* $\sum_{\{i,j\} \in E} a_{ij} y_{ij} \leqslant a_0$ *be a face-defining inequality for* Q^n_T *(for* $\tilde{Q}^n_T$*, respectively), then* $\sum_{i \leqslant j} a_{ij}(x_{ij} + x_{ji}) \leqslant a_0$ *defines a face of* P^n_T *(of* $\tilde{P}^n_T$*, respectively).*

It is clear that any valid inequality $a^T x \leqslant a_0$ for $\tilde{P}^n_T$ with $a_{ij} = a_{ji}$ gives rise to a valid inequality for $\tilde{Q}^n_T$. It is not obvious, however, how to treat valid inequalities $a^T x \leq a_0$ for P^n_T with $a_{ij} \neq a_{ji}$ for some $i \neq j$. A method to symmetrize such inequalities has been investigated by Heller [1956].

Remark 2, however, is important, since we know now that all the valid and facet-defining inequalities for Q^n_T and $\tilde{Q}^n_T$ give rise to valid inequalities for P^n_T and $\tilde{P}^n_T$. Thus we have that the following inequalities are valid for $\tilde{P}^n_T$:

- Subtour elimination constraints; cf. (10), (11).
- 2-matching inequalities; cf. (19).
- Comb inequalities; cf. (34).
- Clique tree inequalities; cf. (33).
- Hypo-Hamiltonian, hypo-semi-Hamiltonian, etc., inequalities; cf. Section 4.4.

An important question now is whether or not the directed versions of inequalities defining facets of Q^n_T (of $\tilde{Q}^n_T$) also define facets of P^n_T (of $\tilde{P}^n_T$). This seems plausible and has been verified – as we shall see – for subtour elimination constraints and some comb inequalities. It is, however, not true for the degree constraints (6) whose directed versions are the sums of two facet-defining directed degree constraints (8).

Moreover, certain directed comb inequalities induce facets of $\tilde{P}^n_T$ but not of P^n_T for small n (cf. the results reported in Section 5.3).

If $a^T y \leq a_0$, $a_0 > 0$, defines a facet of Q^n_T, say, then there are $\frac{1}{2}n(n-1)-n$ linearly independent incidence vectors of undirected tours satisfying the inequality with equality. By directing each of these tours in the two possible ways we obtain $n(n-1)-2n$ directed tours whose incidence vectors satisfy the directed version of $a^T y \leq a_0$ with equality. It is not known under what conditions these incidence vectors are linearly independent. If they are, then these are not enough to prove that the directed version of the inequality defines a facet of P^n_T, since $\dim(P^n_T) = n(n-1)-2n+1$. We have to find one more tour whose incidence vector satisfies the inequality with equality and is linearly independent from the others.

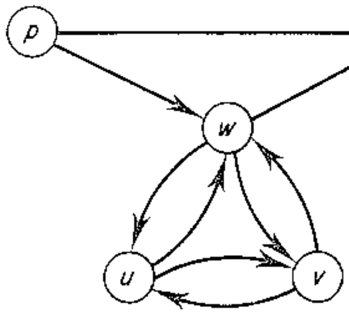
Research problem *Find reasonable sufficient (and necessary) conditions which imply that the directed version of an inequality defining a facet of* Q^n_T *(of* $\tilde{Q}^n_T$*) also defines a facet of* P^n_T *(of* $\tilde{P}^n_T$*).*

There are, of course, valid inequalities for $\tilde{P}^n_T$ which are not symmetric. Interesting classes of such inequalities have been discussed by Grötschel [1977a] and Grötschel & Padberg [1977]. We shall now describe some of these.

Proposition 6 *Let* W *be a vertex set in* $D_n = (V, A)$ *with* $2 \leq |W| = k \leq n-2$, *let* $w \in W$ *and* $p, q \in V - W$, *then*

$$x(A(W)) + x_{pw} + x_{pq} + x_{wq} \leq k$$

is called a T_k*-inequality and is valid with respect to* $\tilde{P}^n_T$.



$x_{uv} + x_{vu} + x_{uw} + x_{wu} + x_{vw}$

$+ x_{wv} + x_{pw} + x_{pq} + x_{wq} \leq 3$

Figure 8.8

A T_3-inequality with $W = \{u, v, w\}$ and the corresponding digraph are shown in Figure 8.8.

T_k-inequalities can be generalized by attaching a source p and a sink q to a comb as follows.

Proposition 7 *Let H be a handle and $T_1, \ldots, T_s$ be teeth satisfying:* (a) $|H \cap T_i| \geq 1$, (b) $|T_i - H| \geq 1$, $i = 1, \ldots, s$; (c) $|T_i \cap T_j| = \emptyset$, $1 \leq i < j \leq s$; *and* (d) $s \geq 3$ *and s odd. Let* $p, q \in V - (H \cup \bigcup_{i=1}^{s} T_i)$, *then*

$$x(A(H)) + \sum_{i=1}^{s} x(A(T_i)) + \sum_{v \in H} (x_{pv} + x_{vq}) + x_{pq}$$

$$\leq |H| + \sum_{i=1}^{s} (|T_i| - 1) - \frac{s+1}{2} + 1 \; (=: s(C) + 1)$$

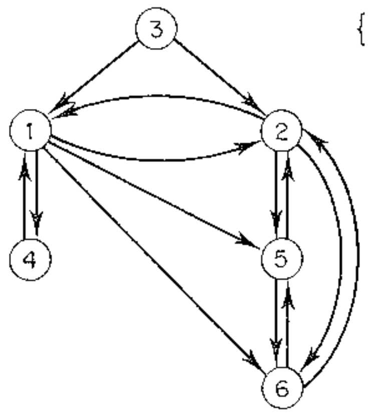
is called a C2-inequality and is valid with respect to $\tilde{P}_T^n$.

Proposition 8 *Let i_1, i_2, i_3 be three different vertices and let W_1, W_2 be subsets of V such that* (a) $W_1 \cap W_2 = \emptyset$, (b) $W_1 \cap \{i_1, i_2, i_3\} = \{i_1\}$, (c) $W_2 \cap \{i_1, i_2, i_3\} = \{i_2\}$, (d) $|W_j| \geq 2$, $j = 1, 2$. *Then*

$$x(A(W_1)) + x(A(W_2)) + \sum_{j \in W_2} x_{i_1 j} + x_{i_2 i_1} + x_{i_3 i_1} + x_{i_3 i_2} \leq |W_1| + |W_2| - 1$$

is called a C3-inequality and is valid with respect to $\tilde{P}_T^n$.

The arcs having positive coefficients in a C3-inequality are shown in Figure 8.9.



$\{i_1, i_2, i_3\} = \{1, 2, 3\}$

$W_1 = \{1, 4\}$

$W_2 = \{2, 5, 6\}$

Figure 8.9

Replacing 'cycle', 'path', 'edge' and 'graph' by 'directed cycle', 'directed path', 'arc' and 'digraph', we obtain the notions of maximal non-Hamiltonian, hypo-Hamiltonian and hypo-semi-Hamiltonian, etc., digraphs. Clearly, if $D=(V, B)$ is a maximal non-Hamiltonian subdigraph of D_n, then

$$x(B) \leq |V|-1$$

is valid with respect to $\tilde{P}^n_T$; and similarly

$$x(B) \leq |V|-2$$

is valid with respect to $\tilde{P}^n_T$ if $D=(V, B)$ is a maximal non-semi-Hamiltonian subdigraph of D_n. The remarks made about such kinds of inequalities in Section 4.4 with respect to the symmetric TSP apply – *mutatis mutandis* – also to the asymmetric TSP. In the asymmetric case there is an example of a maximal hypo-Hamiltonian digraph such that the corresponding inequality does not define a facet of $\tilde{P}^n_T$; cf. Grötschel & Wakabayashi [1981a], and see Exercise 22.

5.3 Facets of P^n_T and $\tilde{P}^n_T$

We now report which of the inequalities introduced in the foregoing sections are known to define facets of P^n_T or $\tilde{P}^n_T$. The results are not as complete as for the symmetric TSP, and many cases are still open. Unless otherwise mentioned, all results are from Grötschel [1977a] and Grötschel & Padberg [1977]. As usual, $D_n=(V, A)$ is the complete digraph on n vertices.

Theorem 23

(a) $\operatorname{aff}(P^n_T)=\operatorname{aff}(P^n_A)=\{x \in \mathbb{R}^A \mid x(\vec{\delta}(i))=1, i=2, \ldots, n, x(\overleftarrow{\delta}(i))=1, i=1, \ldots, n\}$, *and* $\dim(P^n_T)=|A|-2|V|+1$.

(b) *The nonnegativity constraints* $x_{ij} \geq 0$ *define facets of* P^n_T *for all* $(i, j) \in A$, $n \geq 5$ (*cf. Proposition* 4 *and Exercise* 20).

(c) *Let* $n \geq 5$. (*Exercise: How about the case* $n=3, 4$?)

(c$_1$) *The subtour elimination constraints*

$$x(A(W)) \leq |W|-1$$

define facets of P^n_T *if* $2 \leq |W| \leq n-2$.

(c$_2$) *Two different subtour elimination constraints* $x(A(W)) \leq |W|-1$ *and* $x(A(W')) \leq |W'|-1$ *are equivalent with respect to* P^n_T *if and only if* $W'=V-W$.

(c$_3$) *For* $W \subseteq V$, $2 \leq |W| \leq n-2$, *the loop constraints*

$$x(\overleftarrow{\delta}(W))=x(\vec{\delta}(V-W)) \geq 1$$

are equivalent to the subtour elimination constraints with respect to P^n_T.

(d) *The directed versions of the comb inequalities* (34) *are not known to define facets of* P^n_T. *In fact, the directed versions of the* 2-*matching inequalities*

(22) *with three teeth have been shown not to define facets of* P_T^6 *and* P_T^7 *by means of a computer program (cf. Grötschel [1977a]). Nothing is known about general (directed) clique tree inequalities.*

(e) *It is not known which of the C2-inequalities (cf. Proposition 7) define facets of* P_T^n.

(f) *Let* $n \geq 4$, *then a* T_k*-inequality (cf. Proposition 6) defines a facet of* P_T^n *if and only if* $2 \leq k \leq n-2$ *and* $k \neq n-3$. *If* $n \geq 5$, $k \neq n-1$, *then two different facet-defining* T_k*-inequalities define different facets of* P_T^n. *In case* $n = 4$, *the 24 different* T_2*-inequalities define six different facets only.*

(g) *C3-inequalities (Proposition 8) define facets of* P_T^n *if* $|W_1|+|W_2| = n-1$. *All other cases are open.*

(h) *For* $k \leq n-1$ *the* $\vec{D}_k$*- and* $\overleftarrow{D}_k$*-inequalities (cf. Theorem 22) define facets of* P_T^n *in case* $k = 3$ *or* $k = 4$. *All other cases are open.*

There are a few other inequalities known which define facets of P_T^n, e.g. the inequality corresponding to the digraph shown in Figure 8.7(b_4) for $n \geq 5$ (this is an E_4-inequality considered by Grötschel & Padberg [1977]) or some hypo-Hamiltonian inequalities for small n; cf. Grötschel & Wakabayashi [1981a]. But there are no further large classes of facet-defining inequalities known.

By comparing Theorem 23 with the 'nice' relatives of P_T^n described in Theorem 7, we can conclude the following.

Corollary 5 *Let* $n \geq 5$, *then except for the subtour elimination constraint* $x(A(W)) \leq n-2$, $W = \{2, \ldots, n\}$, *each inequality of the complete and nonredundant system of inequalities (26) and (28) defining the facets of the arborescence polytope* P_B^n *and of the antiarborescence polytope* $P_{\bar{B}}^n$ *on* D_n', *respectively, also defines a facet of* P_T^n. *Moreover, this system of inequalities is nonredundant with respect to* P_T^n.

The relation of the assignment polytope P_A^n to P_T^n is obvious.

Not much is known about the completeness of the system defined above for P_T^n, n small. Clearly $P_T^3 = P_A^3$, i.e. the case $n = 3$ is trivial.

The case $n = 4$ seems to be an odd case. Neither the nonnegativity constraints nor the subtour elimination constraints define facets of P_T^4. P_T^4 obviously has six vertices, and is of dimension 5 by Theorem 20. Thus the six incidence vectors of tours form a set of linearly independent points. This implies that every five-element subset of the vertices spans a facet of P_T^4, and hence that P_T^4 has six facets. Recall that the T_2-inequalities (Proposition 6) have the form $x_{i_1i_2} + x_{i_2i_1} + x_{i_3i_1} + x_{i_1i_4} \leq 2$. Thus there are 24 inequalities of this type. It was shown by Grötschel [1977a] that these inequalities define facets of P_T^4 but only six different ones. Hence a system of six T_2-inequalities plus degree equations suffices for a complete and nonredundant description of P_T^4. Such a system describing P_T^4 completely and nonredundantly is given in Table 8.1.

Table 8.1

12	13	14	21	23	24	31	32	34	41	42	43	
1	1	1										$=1$
			1	1	1							$=1$
						1	1	1				$=1$
									1	1	1	$=1$
1							1			1		$=1$
	1			1							1	$=1$
		1			1			1				$=1$
1	1			1	1					1		≤ 2
1	1						1	1			1	≤ 2
	1	1	1	1					1			≤ 2
			1	1		1		1			1	≤ 2
1		1				1	1		1			≤ 2
			1		1	1	1			1		≤ 2

In an abstract, Heller gives a system of 224 equations and inequalities and claims that this completely describes P_T^5 [Heller, 1953]. In another abstract, Kuhn proves that this is wrong and gives a much larger system of 390 equations and inequalities which is claimed to be complete for P_T^5 [Kuhn, 1955a; Gomory, 1966].

Let us now turn to the monotone asymmetric traveling salesman polytope $\tilde{P}_T^n$ and review the results known about facet-defining inequalities. The results mentioned in the next theorem are from Grötschel [1977a].

Theorem 24

(a) *For all $n \geq 3$ all nonnegativity constraints $x_{ij} \geq 0$, $(i, j) \in A$, define facets of $\tilde{P}_T^n$.*

(b) *For all $n \geq 3$ the degree constraints $x(\overleftarrow{\delta}(v)) \leq 1$ and $x(\vec{\delta}(v)) \leq 1$, $v \in V$, define facets of $\tilde{P}_T^n$.*

(c) *For all $n \geq 3$ the subtour elimination constraints $x(A(W)) \leq |W| - 1$ define facets of $\tilde{P}_T^n$ if and only if $2 \leq |W| \leq n-1$.*

(d) *All inequalities obtained from lifting cycle constraints $x(C) \leq |C| - 1$, $2 \leq |C| \leq n-1$, sequentially (in any order) define facets of $\tilde{P}_T^n$. In particular, all $\overleftarrow{D}_k$- and $\vec{D}_k$-inequalities, $3 \leq k \leq n-1$, define facets of $\tilde{P}_T^n$.*

(e) *All directed versions of Chvátal comb inequalities (cf. (34)), such that the corresponding comb has three teeth, define facets of P_T^n for $n \geq 6$. All other cases are open.*

(f) *All T_k-inequalities, $2 \leq k \leq n-2$, define facets of $\tilde{P}_T^n$, $n \geq 4$. No other C2-inequalities are known to define facets.*

(g) *The C3-inequalities from Proposition 8 define facets if $|W_2| = 2$ and $2 \leq |W_1| \leq n-3$, $n \geq 5$. All other cases are open.*

Again we can compare Theorem 24 with the nice relatives of $\tilde{P}_T^n$ described in Theorem 6 (identifying vertex $n+1$ of D_n' with vertex 1 in D_n).

Corollary 6 *Let* $n \geq 3$, *then each of the inequalities* (26), (27), (28), (29) *(providing a complete and nonredundant system for* $\tilde{P}_{\bar{B}}^n$ *and* $\tilde{P}_B^n$, *respectively) defines a facet of* $\tilde{P}_T^n$. *Moreover, the system of these inequalities is nonredundant with respect to* $\tilde{P}_T^n$.

Thus $\tilde{P}_T^n$ inherits all facets of $\tilde{P}_{\bar{B}}^n$ and $\tilde{P}_B^n$, while P_T^n inherits all facets of $P_{\bar{B}}^n$ and P_B^n but one.

The proofs of the above results are quite complicated and use a rather involved technical machinery.

To close this section we would like to mention some 'bad' facets (cf. Section 4.4) of $\tilde{P}_T^n$ which were found by Grötschel & Wakabayashi [1981a, 1981b].

Clearly, replacing an edge $\{i, j\}$ of a hypo-Hamiltonian or hypo-semi-Hamiltonian graph by two arcs (i, j) and (j, i), one obtains a hypo-Hamiltonian or hypo-semi-Hamiltonian digraph. In fact, there are many more constructions of such digraphs. Hypo-Hamiltonian (hypo-semi-Hamiltonian) digraphs, for example, are known to exist for any order $n \geq 6$ ($n \geq 7$).

We already know that there are maximal hypo-Hamiltonian digraphs which do not define facets of $\tilde{P}_T^n$; cf. Exercise 22. However, it was shown by Grötschel & Wakabayashi [1981a, 1981b] that almost all known maximal hypo-Hamiltonian and hypo-semi-Hamiltonian digraphs (and these are quite a lot) define facets of $\tilde{P}_T^n$. In particular, every polytope $\tilde{P}_T^n$, $n \geq 7$, has some hypo-Hamiltonian and hypo-semi-Hamiltonian facets. We give two examples.

The digraph $D = (V, B)$ shown in Figure 8.10 is maximal hypo-semi-

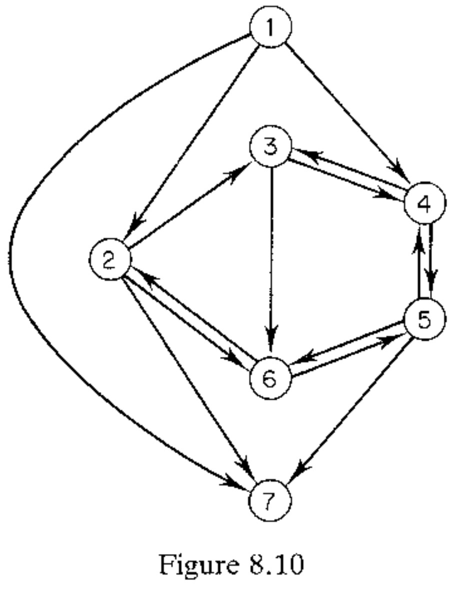


Figure 8.10

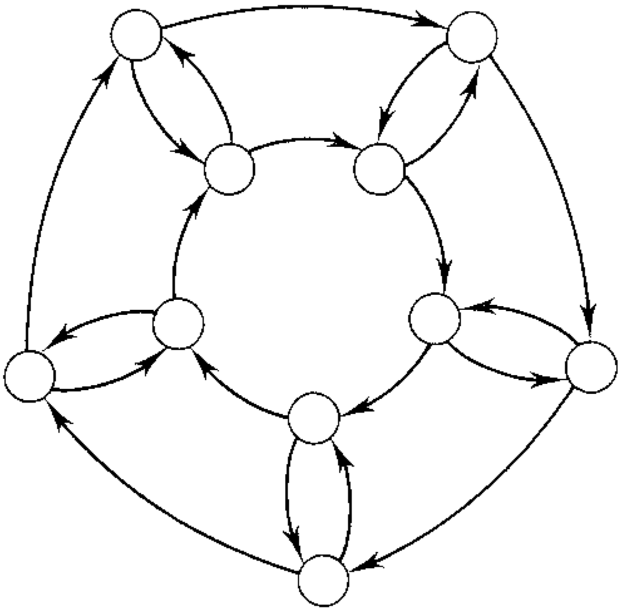


Figure 8.11

Hamiltonian and the inequality $x(B) \leqslant 5$ defines a facet of $\tilde{P}_T^n$ for all $n \geqslant 7$. In fact, $\tilde{P}_T^n$ has $5040 \binom{n}{7}$ facets of this type.

The digraph $D' = (V, B')$ shown in Figure 8.11 is hypo-Hamiltonian, but not maximal. D' has the property that every maximal hypo-Hamiltonian digraph $D = (V, B)$ with $B' \subseteq B$ defines a facet $x(B) \leqslant 9$ of $\tilde{P}_T^{10}$, but not of $\tilde{P}_T^n$, $n \neq 10$.

Finally, we would like to mention a peculiar case. Consider the digraph D on the vertices $V = \{1, 2, \ldots, 6\}$ with arc set $B \cup C_1 \cup C_2 \cup \{(1, 3)\}$, where $B = \{(i, i+3), (i+3, i) \mid i = 1, 2, 3\}$, $C_1 = \{(1, 2), (2, 3), (3, 1)\}$, $C_2 = \{(4, 5), (5, 6), (6, 4)\}$. This digraph D is the digraph shown in Figure 8.12 without arc $(4, 6)$. D is hypo-Hamiltonian but not maximal. It was shown by

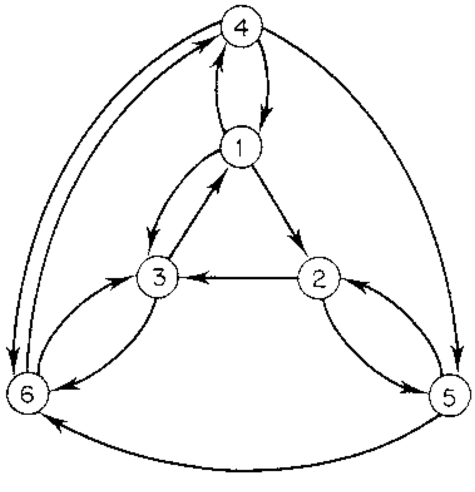


Figure 8.12

Grötschel & Wakabayashi [1981a] that the inequality

$$x(C_2)+2x(B)+3x(C_1)+4x_{13}\leqslant 10$$

defines a facet of $\tilde{P}_T^6$. By sequential lifting, it is easy to see that the inequality also defines a facet of $\tilde{P}_T^n, n\geqslant 6$. This is a strange case where a hypo-Hamiltonian digraph induces a facet, but where the arcs in the corresponding inequality have to be weighted. So, even for $n=6$ there are facets such that the corresponding inequalities have coefficients not just 0, 1 or 2.

5.4 Neighbor relations

Up to now we have looked at the traveling salesman polyhedra from a facet point of view; i.e., we have tried to find inequalities which define maximal faces, which are necessary in any complete description, and which can be used in linear programming based cutting plane procedures for the TSP.

There is another more combinatorial aspect, related to minimal faces, which could also be useful in linear programming approaches. Namely, the simplex method has the property that it starts at a vertex of a polytope P and (except for degenerate pivot steps) moves to vertices $v_2, v_3, \ldots, v_k$ of P such that two successive vertices v_i and v_{i+1}, $i=1,\ldots,k-1$, are adjacent on P. Here *adjacency* means that v_i and v_{i+1} belong to a common face of dimension one of P, which is usually called an *edge* of P. If one could get handy descriptions of adjacency, one might be able to specialize the simplex method for certain polytopes in a combinatorial fashion and derive efficient algorithms this way.

There are quite a few interesting theoretical results known about adjacency of vertices on polytopes associated with combinatorial optimization problems (see Hausmann [1980] for an extensive survey), but up to now these results have not found algorithmic applications. Nevertheless, there is some hope that studies concerning adjacency may lead to new types of algorithms, or to improvements of existing ones. This area remains to be explored.

Almost all studies of neighbor relations made so far have been concerned with the asymmetric traveling salesman polytope P_T^n. To make the approach clear, we will introduce the necessary concepts in full generality.

Suppose P is a polytope with vertex set V. Let $G=(V,E)$ be the undirected graph whose vertices are the vertices of P. Two vertices of G are adjacent if and only if (considered as vertices of P) they are adjacent on P, i.e. $\{v, w\}$ is an edge of G if and only if there is an edge of P which contains v and w. (As one can see from the terminology, graphs associated with polyhedra are one of the sources of graph theory.) Let us call $G=(V,E)$ the *skeleton* of P.

There is a large body of literature about the skeletons of polytopes, in particular about the skeletons of 3-dimensional polytopes; see for example, Grünbaum [1967] for related references. One may ask the following questions: Is the skeleton of a certain polytope Hamiltonian; what is its connectivity, etc.? Of particular importance are mutual characterizations of the following type. Is there a 'nice' graphical characterization of adjacency on P, or is there a 'nice' polyhedral characterization of adjacency in the skeleton G?

One important linear programming related parameter is the diameter of G and P. Let us define the *distance* of two vertices v and w of G, denoted by $\mathrm{dist}(v, w)$, as the length of the shortest path from v to w in G. For instance, adjacent vertices have distance 1. The *diameter* of G, $\mathrm{diam}(G)$, is the maximum distance of any pair of vertices of G, i.e.

$$\mathrm{diam}(G) := \max_{u,v \in V} \{\mathrm{dist}(v, w)\}.$$

If G is the skeleton of a polytope P, then $\mathrm{diam}(G)$ is also called the *diameter* of P.

The polytope whose skeleton has probably been studied most intensively is the assignment polytope (see for instance Balinski & Russakoff [1974]; Hausmann [1980], Heller [1955, 1956] and Padberg & Rao [1974]). The polytope considered in these papers is not exactly our polytope P^n_{A} (cf. Theorem 5), but a slightly larger one which we will denote by $\bar{P}^n_{\mathrm{A}}$ for convenience:

$$\bar{P}^n_{\mathrm{A}} = \left\{ x \in \mathbb{R}^{n^2} \mid x_{ij} \geq 0,\ i, j = 1, \ldots, n;\ \sum_{i=1}^{n} x_{ij} = 1,\ j = 1, \ldots, n; \right.$$
$$\left. \sum_{j=1}^{n} x_{ij} = 1,\ i = 1, \ldots, n \right\}$$

In other words, in the assignment polytope $\bar{P}^n_{\mathrm{A}}$, loops (i.e. assignments (i, i)) are allowed; which is not the case in P^n_{A}. The polytope P^n_{A} can be obtained from $\bar{P}^n_{\mathrm{A}}$ by setting $x_{ii} = 0$, $i = 1, \ldots, n$. Each vertex of the assignment polytope $\bar{P}^n_{\mathrm{A}}$ corresponds to a permutation of $\{1, \ldots, n\}$; or in graphical terms, each vertex corresponds to an arc set which is the union of directed cycles such that no two cycles have a vertex in common (here loops are considered as directed cycles of length 1).

Now let G^n_{A} denote the skeleton of $\bar{P}^n_{\mathrm{A}}$. The following adjacency characterization for G^n_{A} and $\bar{P}^n_{\mathrm{A}}$ is due to Balinski & Russakoff [1974].

A *directed trail* is a sequence $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ of distinct arcs, and is called *closed* if $v_1 = v_k$.

Theorem 25 *Let v, w be two different vertices of $\bar{P}^n_{\mathrm{A}}$, $n \geq 3$, and let A_v and A_w denote the arc sets (unions of directed cycles) corresponding to v and w. Then v and w are adjacent on $\bar{P}^n_{\mathrm{A}}$ if and only if $A_v \cup A_w$ is a closed directed trail.*

Another example of such an adjacency characterization – now permutation oriented – is due to Padberg & Rao [1974]. If v is a vertex of $\bar{P}^n_{\mathrm{A}}$, then π_v denotes the permutation of $\{1, \ldots, n\}$ corresponding to v, and by '$\circ$' we denote the usual composition of permutations.

Theorem 26 *Let v, w be two different vertices of $\bar{P}^n_{\mathrm{A}}$, $n \geq 3$, then v and w are adjacent on $\bar{P}^n_{\mathrm{A}}$ if and only if there exists a permutation π such that $\pi_v = \pi_w \circ \pi$ holds and π consists of a single cyclic permutation of length greater than 1 and possibly some cycles of length 1.*

In algebra it is well known that given any permutation π, one can generate a sequence $\pi_0 = \pi, \pi_1, \ldots, \pi_{n!} = \pi$ of permutations such that all permutations $\pi_1, \ldots, \pi_{n!}$ are distinct and $\pi_{i+1} = \pi_i \circ \tau$, where τ is a transposition. By Theorem 26, π_{i+1} and π_i are adjacent; thus we can conclude the following result.

Theorem 27 *The skeleton G^n_{A} of $\bar{P}^n_{\mathrm{A}}$ is Hamiltonian, $n \geq 3$.*

An interesting result of Heller [1955] is the following.

Theorem 28 *If v is the vertex of $\bar{P}^n_{\mathrm{A}}$ corresponding to the identity permutation, then the set of neighbors of v on P^n_{A} is precisely the set of incidence vectors of tours.*

With respect to P^n_{T}, no good necessary and sufficient criteria for adjacency are known. The following sufficient condition was given by Murty [1969].

Theorem 29 *Let S and T be two different tours in D_n. Then the incidence vectors x^S and x^T are adjacent on P^n_{T} if there is no other tour R such that $S \cap T \subseteq R \subseteq S \cup T$.*

Murty claimed that this condition is also necessary, but this was shown to be wrong; see Rao [1976], who also gave sufficient conditions for adjacency on P^n_{T} other than Theorem 29. Coloring criteria for general 0–1-polytopes and their relationship to the adjacency of tours of the traveling salesman are extensively discussed by Hausmann [1980]; see also Balas & Padberg [1979]. These latter conditions are often helpful to prove adjacency in practice.

On the other hand, it is quite unlikely that a 'good' characterization of adjacency on P^n_{T} or Q^n_{T} can ever be found. Papadimitriou [1978] has proved that the question 'Are two given vertices of P^n_{T}, of Q^n_{T} respectively, nonadjacent?' is $\mathcal{NP}$-complete.

The following theorem – based on a further adjacency criterion – is one of the deepest adjacency results for P^n_{T} and quite surprising. It is due to Padberg & Rao [1974].

Theorem 30 *The diameter of P^n_{T} is equal to 2 for $n \geq 6$, and equal to 1 for $3 \leq n \leq 5$.*

Geometrically interpreted, Theorem 30 states the following. Suppose two persons A and B are sitting on vertices of P^n_{T}. Person A wants to walk to person B, where walking is done by making a step from one vertex to an adjacent vertex (along an edge of the polytope). Then it is possible for A to reach B in at most two steps.

This does not mean, however, that the simplex method could get from a given basic solution to an optimal basic solution in two pivot steps by some proper column-and-row selection rule. Indeed, quite a large number of degenerate pivots (corresponding to one and the same vertex) may be required to get from one vertex to an optimal one. But Padberg & Rao [1974] have also shown that – if columns and rows are chosen properly – the number of necessary pivot steps is not too large.

Theorem 31 *Let v_1, v_2 be two vertices of P^n_T and let B_1 denote a basis corresponding to v_1. Then there is a sequence of pivot operations such that starting from B_1, a basis corresponding to v_2 is reached after at most $2n-1$ steps.*

So if we knew how to select rows and columns properly we could solve TSPs in a few simplex steps. A direct consequence of Theorem 31 is that the famous Hirsch conjecture [Dantzig, 1963] holds for P^n_T.

To our knowledge, nothing similar is known with respect to Q^n_T. Thus we can close this section with a few research problems.

Research problems

(a) *Determine the diameter of Q^n_T. (Conjecture:* $\mathrm{diam}(Q^n_T)=2$.)

(b) *Is the skeleton of Q^n_T Hamiltonian?*

(c) *What are the diameters of $\tilde{Q}^n_T$ and $\tilde{P}^n_T$?*

Exercises

19. Prove that the incidence vectors of the $|A|-2|V|+2$ directed tours obtained as outlined in the text following Theorem 20 are linearly independent.

20. (a) Prove that the nonnegativity constraints define facets of P^3_T, but only two different ones.

(b) Prove that every nonnegativity constraint defines a face of P^4_T which is the intersection of two unique facets of P^4_T.

21. Let C be a cycle of length $3 \leq k \leq n-1$ in D_n. Let (i, j) be a diagonal of C. Prove that the coefficient a_{ij} obtained for (i, j) by the formula of Theorem 21 is 2.

22. (a) Prove that the digraph $D=(V, B)$ shown in Figure 8.12 is a maximal hypo-Hamiltonian digraph.

(b) Prove that the hypo-Hamiltonian inequality $n(B) \leq 5$ corresponding to D does not define a facet of $\tilde{P}^6_T$. (In fact, the face defined by this inequality has dimension 28. To be a facet it should have dimension 29.)