4 Euler Tours and Hamilton Cycles

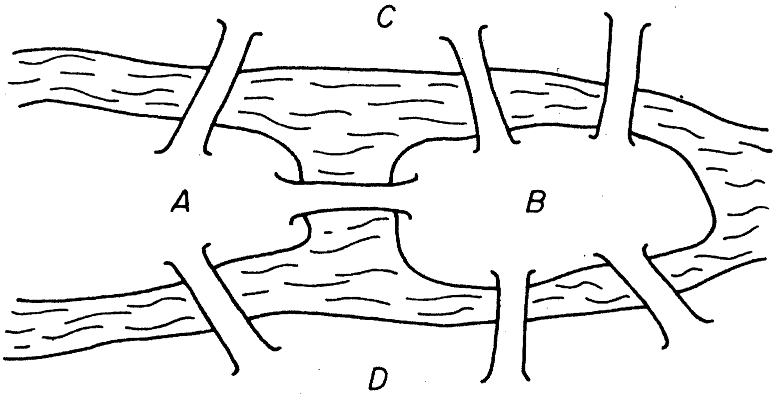
4.1 EULER TOURS

A trail that traverses every edge of G is called an *Euler trail* of G because Euler was the first to investigate the existence of such trails in graphs. In the earliest known paper on graph theory (Euler, 1736), he showed that it was impossible to cross each of the seven bridges of Königsberg once and only once during a walk through the town. A plan of Königsberg and the river Pregel is shown in figure 4.1*a*. As can be seen, proving that such a walk is impossible amounts to showing that the graph of figure 4.1*b* contains no Euler trail.

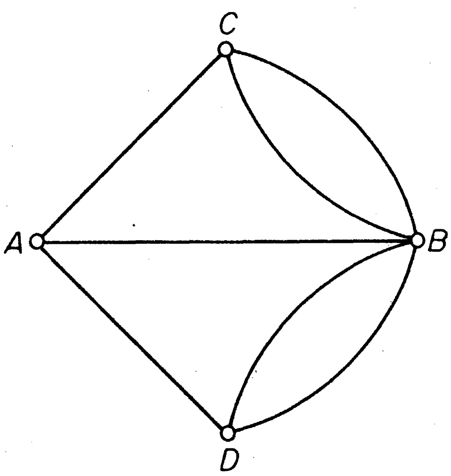
A *tour* of G is a closed walk that traverses each edge of G at least once. An *Euler tour* is a tour which traverses each edge exactly once (in other words, a closed Euler trail). A graph is *eulerian* if it contains an Euler tour.

Theorem 4.1 A nonempty connected graph is eulerian if and only if it has no vertices of odd degree.

Proof Let G be eulerian, and let C be an Euler tour of G with origin (and terminus) u. Each time a vertex v occurs as an internal vertex of C, two of the edges incident with v are accounted for. Since an Euler tour contains



(a)



(b)

Figure 4.1. The bridges of Königsberg and their graph

every edge of G, $d(v)$ is even for all $v \neq u$. Similarly, since C starts and ends at u, $d(u)$ is also even. Thus G has no vertices of odd degree.

Conversely, suppose that G is a noneulerian connected graph with at least one edge and no vertices of odd degree. Choose such a graph G with as few edges as possible. Since each vertex of G has degree at least two, G contains a closed trail (exercise 1.7.2). Let C be a closed trail of maximum possible length in G. By assumption, C is not an Euler tour of G and so $G-E(C)$ has some component G' with $\varepsilon(G')>0$. Since C is itself eulerian, it has no vertices of odd degree; thus the connected graph G' also has no vertices of odd degree. Since $\varepsilon(G')<\varepsilon(G)$, it follows from the choice of G that G' has an Euler tour C'. Now, because G is connected, there is a vertex v in $V(C) \cap V(C')$, and we may assume, without loss of generality, that v is the origin and terminus of both C and C'. But then CC' is a closed trail of G with $\varepsilon(CC')>\varepsilon(C)$, contradicting the choice of C □

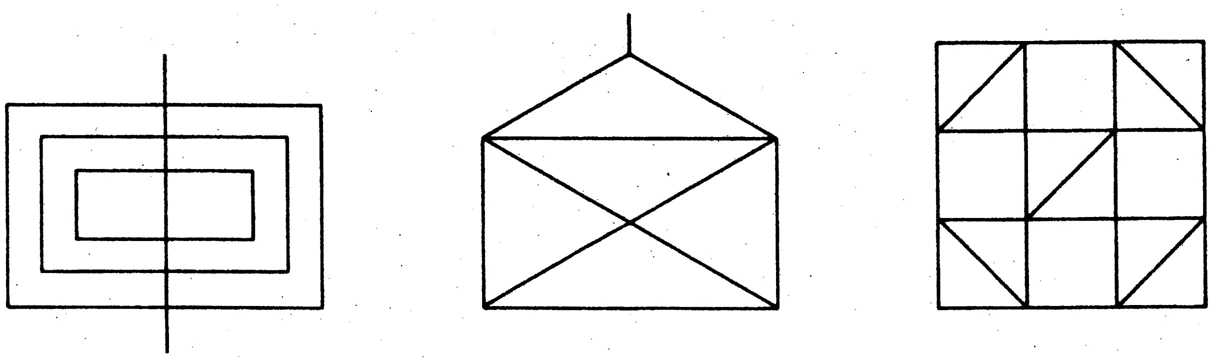
Corollary 4.1 A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.

Proof If G has an Euler trail then, as in the proof of theorem 4.1, each vertex other than the origin and terminus of this trail has even degree.

Conversely, suppose that G is a nontrivial connected graph with at most two vertices of odd degree. If G has no such vertices then, by theorem 4.1, G has a closed Euler trail. Otherwise, G has exactly two vertices, u and v, of odd degree. In this case, let $G+e$ denote the graph obtained from G by the addition of a new edge e joining u and v. Clearly, each vertex of $G+e$ has even degree and so, by theorem 4.1, $G+e$ has an Euler tour $C = v_0e_1v_1 \ldots e_{\varepsilon+1}v_{\varepsilon+1}$, where $e_1 = e$. The trail $v_1e_2v_2 \ldots e_{\varepsilon+1}v_{\varepsilon+1}$ is an Euler trail of G □

Exercises

4.1.1 Which of the following figures can be drawn without lifting one's pen from the paper or covering a line more than once?



4.1.2 If possible, draw an eulerian graph G with ν even and ε odd; otherwise, explain why there is no such graph.

4.1.3 Show that if G is eulerian, then every block of G is eulerian.

4.1.4 Show that if G has no vertices of odd degree, then there are edge-disjoint cycles $C_1, C_2, \ldots, C_m$ such that $E(G) = E(C_1) \cup E(C_2) \cup \ldots \cup E(C_m)$.

4.1.5 Show that if a connected graph G has $2k > 0$ vertices of odd degree, then there are k edge-disjoint trails $Q_1, Q_2, \ldots, Q_k$ in G such that $E(G) = E(Q_1) \cup E(Q_2) \cup \ldots \cup E(Q_k)$.

4.1.6* Let G be nontrivial and eulerian, and let $v \in V$. Show that every trail of G with origin v can be extended to an Euler tour of G if and only if $G - v$ is a forest. (O. Ore)

4.2 HAMILTON CYCLES

A path that contains every vertex of G is called a *Hamilton path* of G; similarly, a *Hamilton cycle* of G is a cycle that contains every vertex of G. Such paths and cycles are named after Hamilton (1856), who described, in a letter to his friend Graves, a mathematical game on the dodecahedron (figure 4.2*a*) in which one person sticks five pins in any five consecutive vertices and the other is required to complete the path so formed to a

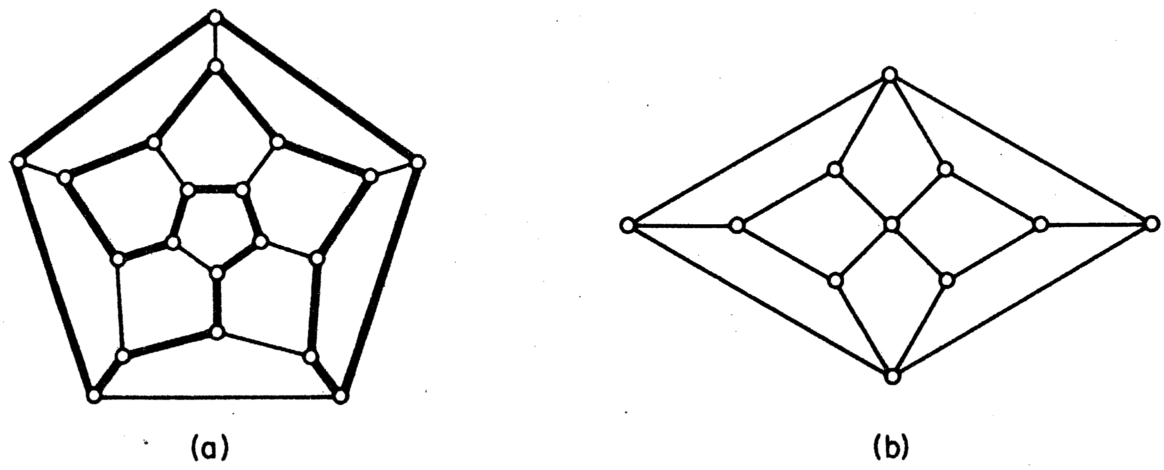


Figure 4.2. (*a*) The dodecahedron; (*b*) the Herschel graph

spanning cycle. A graph is *hamiltonian* if it contains a Hamilton cycle. The dodecahedron is hamiltonian (see figure 4.2*a*); the Herschel graph (figure 4.2*b*) is nonhamiltonian, because it is bipartite and has an odd number of vertices.

In contrast with the case of eulerian graphs, no nontrivial necessary and sufficient condition for a graph to be hamiltonian is known; in fact, the problem of finding such a condition is one of the main unsolved problems of graph theory.

We shall first present a simple, but useful, necessary condition.

Theorem 4.2 If G is hamiltonian then, for every nonempty proper subset S of V

$$\omega(G - S) \leq |S| \tag{4.1}$$

Proof Let C be a Hamilton cycle of G. Then, for every nonempty proper subset S of V

$$\omega(C-S)\leq|S|$$

Also, $C-S$ is a spanning subgraph of $G-S$ and so

$$\omega(G-S)\leq\omega(C-S)$$

The theorem follows □

As an illustration of the above theorem, consider the graph of figure 4.3. This graph has nine vertices; on deleting the three indicated in black, four components remain. Therefore (4.1) is not satisfied and it follows from theorem 4.2 that the graph is nonhamiltonian.

We thus see that theorem 4.2 can sometimes be applied to show that a particular graph is nonhamiltonian. However, this method does not always

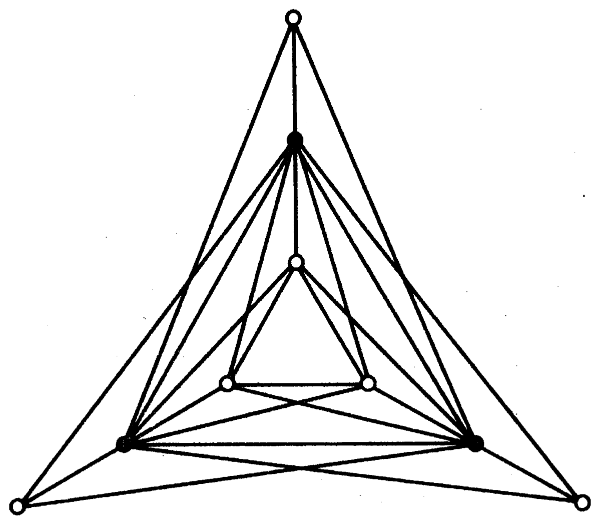


Figure 4.3

work; for instance, the Petersen graph (figure 4.4) is nonhamiltonian, but one cannot deduce this by using theorem 4.2.

We now discuss sufficient conditions for a graph G to be hamiltonian; since a graph is hamiltonian if and only if its underlying simple graph is hamiltonian, it suffices to limit our discussion to simple graphs. We start with a result due to Dirac (1952).

Theorem 4.3 If G is a simple graph with $\nu\geq 3$ and $\delta\geq\nu/2$, then G is hamiltonian.

Proof By contradiction. Suppose that the theorem is false, and let G be a maximal nonhamiltonian simple graph with $\nu\geq 3$ and $\delta\geq\nu/2$. Since $\nu\geq 3$, G cannot be complete. Let u and v be nonadjacent vertices in G. By the choice of G, $G+uv$ is hamiltonian. Moreover, since G is nonhamiltonian,

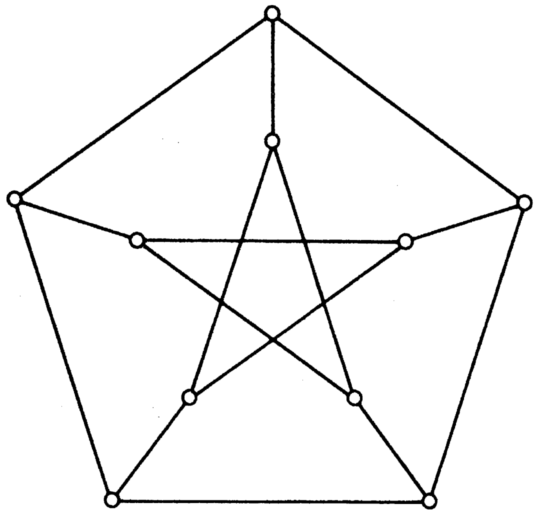


Figure 4.4. The Petersen graph

each Hamilton cycle of $G+uv$ must contain the edge uv. Thus there is a Hamilton path $v_1v_2 \dots v_\nu$ in G with origin $u = v_1$ and terminus $v = v_\nu$. Set

$$S = \{v_i \mid uv_{i+1} \in E\} \quad \text{and} \quad T = \{v_i \mid v_iv \in E\}$$

Since $v_\nu \notin S \cup T$ we have

$$|S \cup T| < \nu \tag{4.2}$$

Furthermore

$$|S \cap T| = 0 \tag{4.3}$$

since if $S \cap T$ contained some vertex v_i, then G would have the Hamilton cycle $v_1v_2 \dots v_iv_\nu v_{\nu-1} \dots v_{i+1}v_1$, contrary to assumption (see figure 4.5).

Using (4.2) and (4.3) we obtain

$$d(u)+d(v) = |S|+|T| = |S \cup T| + |S \cap T| < \nu \tag{4.4}$$

But this contradicts the hypothesis that $\delta \geq \nu/2$ □

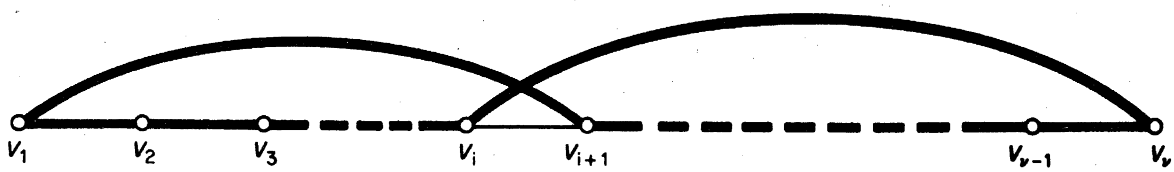


Figure 4.5

Bondy and Chvátal (1974) observed that the proof of theorem 4.3 can be modified to yield stronger sufficient conditions than that obtained by Dirac. The basis of their approach is the following lemma.

Lemma 4.4.1 Let G be a simple graph and let u and v be nonadjacent vertices in G such that

$$d(u)+d(v) \geq \nu \tag{4.5}$$

Then G is hamiltonian if and only if $G+uv$ is hamiltonian.

Proof If G is hamiltonian then, trivially, so too is $G+uv$. Conversely, suppose that $G+uv$ is hamiltonian but G is not. Then, as in the proof of theorem 4.3, we obtain (4.4). But this contradicts hypothesis (4.5) □

Lemma 4.4.1 motivates the following definition. The *closure* of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least ν until no such pair remains. We denote the closure of G by $c(G)$.

Lemma 4.4.2 $c(G)$ is well defined.

Proof Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least ν until no such pair remains. Denote by $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$ the sequences of edges added to G in obtaining G_1 and G_2, respectively. We shall show that each e_i is an edge of G_2 and each f_j is an edge of G_1.

If possible, let $e_{k+1}=uv$ be the first edge in the sequence $e_1, e_2, \ldots, e_n$ that is not an edge of G_2. Set $H=G+\{e_1, e_2, \ldots, e_k\}$. It follows from the definition of G_1 that

$$d_H(u)+d_H(v)\geq \nu$$

By the choice of e_{k+1}, H is a subgraph of G_2. Therefore

$$d_{G_2}(u)+d_{G_2}(v)\geq \nu$$

This is a contradiction, since u and v are nonadjacent in G_2. Therefore each e_i is an edge of G_2 and, similarly, each f_j is an edge of G_1. Hence $G_1=G_2$, and $c(G)$ is well defined □

Figure 4.6 illustrates the construction of the closure of a graph G on six vertices. It so happens that in this example $c(G)$ is complete; note, however, that this is by no means always the case.

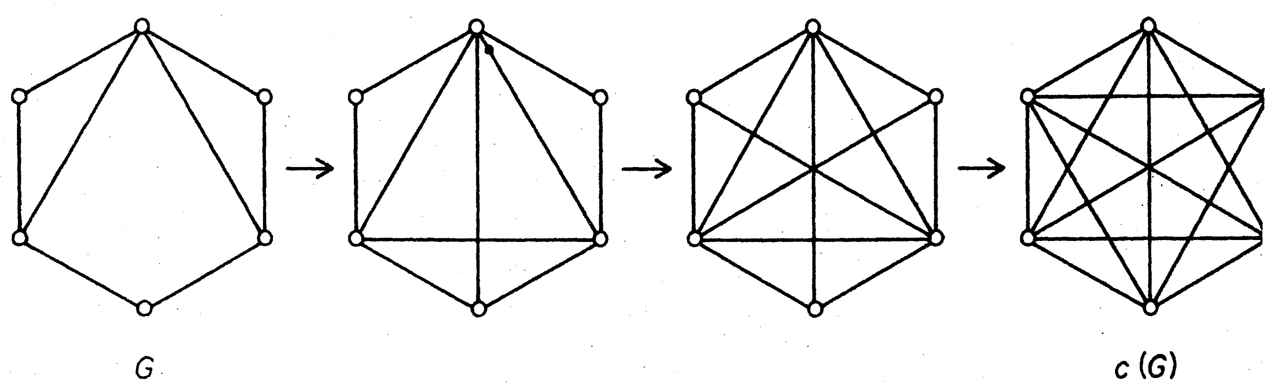


Figure 4.6. The closure of a graph

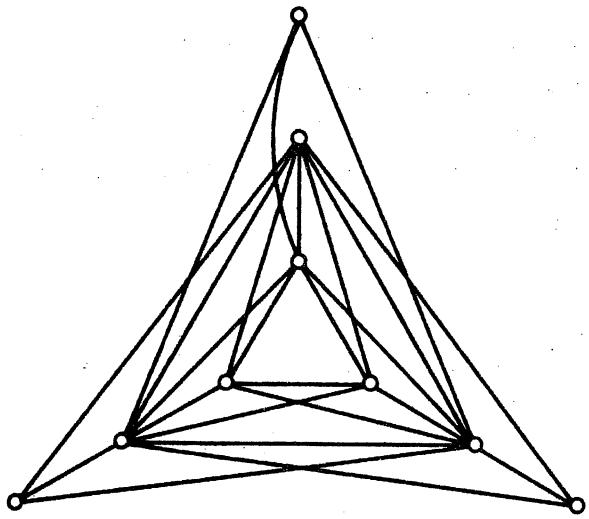


Figure 4.7. A hamiltonian graph

Theorem 4.4 A simple graph is hamiltonian if and only if its closure is hamiltonian.

Proof Apply lemma 4.4.1 each time an edge is added in the formation of the closure □

Theorem 4.4 has a number of interesting consequences. First, upon making the trivial observation that all complete graphs on at least three vertices are hamiltonian, we obtain the following result.

Corollary 4.4 Let G be a simple graph with $\nu \geq 3$. If $c(G)$ is complete, then G is hamiltonian.

Consider, for example, the graph of figure 4.7. One readily checks that its closure is complete. Therefore, by corollary 4.4, it is hamiltonian. It is perhaps interesting to note that the graph of figure 4.7 can be obtained from the graph of figure 4.3 by altering just one end of one edge, and yet we have results (corollary 4.4 and theorem 4.2) which tell us that this one is hamiltonian whereas the other is not.

Corollary 4.4 can be used to deduce various sufficient conditions for a graph to be hamiltonian in terms of its vertex degrees. For example, since $c(G)$ is clearly complete when $\delta \geq \nu/2$, Dirac's condition (theorem 4.3) is an immediate corollary. A more general condition than that of Dirac was obtained by Chvátal (1972).

Theorem 4.5 Let G be a simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3$. Suppose that there is no value of m less than $\nu/2$ for which $d_m \leq m$ and $d_{\nu-m} < \nu - m$. Then G is hamiltonian.

Proof Let G satisfy the hypothesis of the theorem. We shall show that its closure $c(G)$ is complete, and the conclusion will then follow from corollary 4.4. We denote the degree of a vertex v in $c(G)$ by $d'(v)$.

Assume that $c(G)$ is not complete, and let u and v be two nonadjacent vertices in $c(G)$ with

$$d'(u) \leq d'(v) \tag{4.6}$$

and $d'(u)+d'(v)$ as large as possible; since no two nonadjacent vertices in $c(G)$ can have degree sum ν or more, we have

$$d'(u)+d'(v) < \nu \tag{4.7}$$

Now denote by S the set of vertices in $V\backslash\{v\}$ which are nonadjacent to v in $c(G)$, and by T the set of vertices in $V\backslash\{u\}$ which are nonadjacent to u in $c(G)$. Clearly

$$|S| = \nu - 1 - d'(v) \quad \text{and} \quad |T| = \nu - 1 - d'(u) \tag{4.8}$$

Furthermore, by the choice of u and v, each vertex in S has degree at most $d'(u)$ and each vertex in $T \cup \{u\}$ has degree at most $d'(v)$. Setting $d'(u) = m$ and using (4.7) and (4.8), we find that $c(G)$ has at least m vertices of degree at most m and at least $\nu - m$ vertices of degree less than $\nu - m$. Because G is a spanning subgraph of $c(G)$, the same is true of G; therefore $d_m \leq m$ and $d_{\nu-m} < \nu - m$. But this is contrary to hypothesis since, by (4.6) and (4.7), $m < \nu/2$. We conclude that $c(G)$ is indeed complete and hence, by corollary 4.4, that G is hamiltonian □

One can often deduce that a given graph is hamiltonian simply by computing its degree sequence and applying theorem 4.5. This method works with the graph of figure 4.7 but not with the graph G of figure 4.6, even though the closure of the latter graph is complete. From these examples, we see that theorem 4.5 is stronger than theorem 4.3 but not as strong as corollary 4.4.

A sequence of real numbers $(p_1, p_2, \ldots, p_n)$ is said to be *majorised* by another such sequence $(q_1, q_2, \ldots, q_n)$ if $p_i \leq q_i$ for $1 \leq i \leq n$. A graph G is *degree-majorised* by a graph H if $\nu(G) = \nu(H)$ and the nondecreasing degree sequence of G is majorised by that of H. For instance, the 5-cycle is degree-majorised by $K_{2,3}$ because (2, 2, 2, 2, 2) is majorised by (2, 2, 2, 3, 3). The family of degree-maximal nonhamiltonian graphs (those that are degree-majorised by no others) admits of a simple description. We first introduce the notion of the join of two graphs. The *join* $G \vee H$ of disjoint graphs G and H is the graph obtained from $G+H$ by joining each vertex of G to each vertex of H; it is represented diagrammatically as in figure 4.8.

Now, for $1 \leq m < n/2$, let $C_{m,n}$ denote the graph $K_m \vee (K_m^c + K_{n-2m})$, depicted in figure 4.9*a*; two specific examples, $C_{1,5}$ and $C_{2,5}$, are shown in figures 4.9*b* and 4.9*c*.

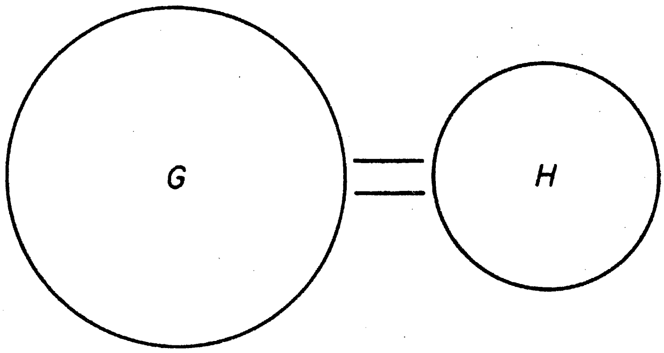


Figure 4.8. The join of G and H

That $C_{m,n}$ is nonhamiltonian follows immediately from theorem 4.2; for if S denotes the set of m vertices of degree $n-1$ in $C_{m,n}$, we have $\omega(C_{m,n}-S)=m+1>|S|$.

Theorem 4.6 (Chvátal, 1972) If G is a nonhamiltonian simple graph with $\nu\geq 3$, then G is degree-majorised by some $C_{m,\nu}$.

Proof Let G be a nonhamiltonian simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1\leq d_2\leq \ldots \leq d_\nu$ and $\nu\geq 3$. Then, by theorem 4.5, there exists $m<\nu/2$ such that $d_m\leq m$ and $d_{\nu-m}<\nu-m$. Therefore $(d_1, d_2, \ldots, d_\nu)$ is majorised by the sequence

$$(m, \ldots, m, \nu-m-1, \ldots, \nu-m-1, \nu-1, \ldots, \nu-1)$$

with m terms equal to m, $\nu-2m$ terms equal to $\nu-m-1$ and m terms equal to $\nu-1$, and this latter sequence is the degree sequence of $C_{m,\nu}$ □

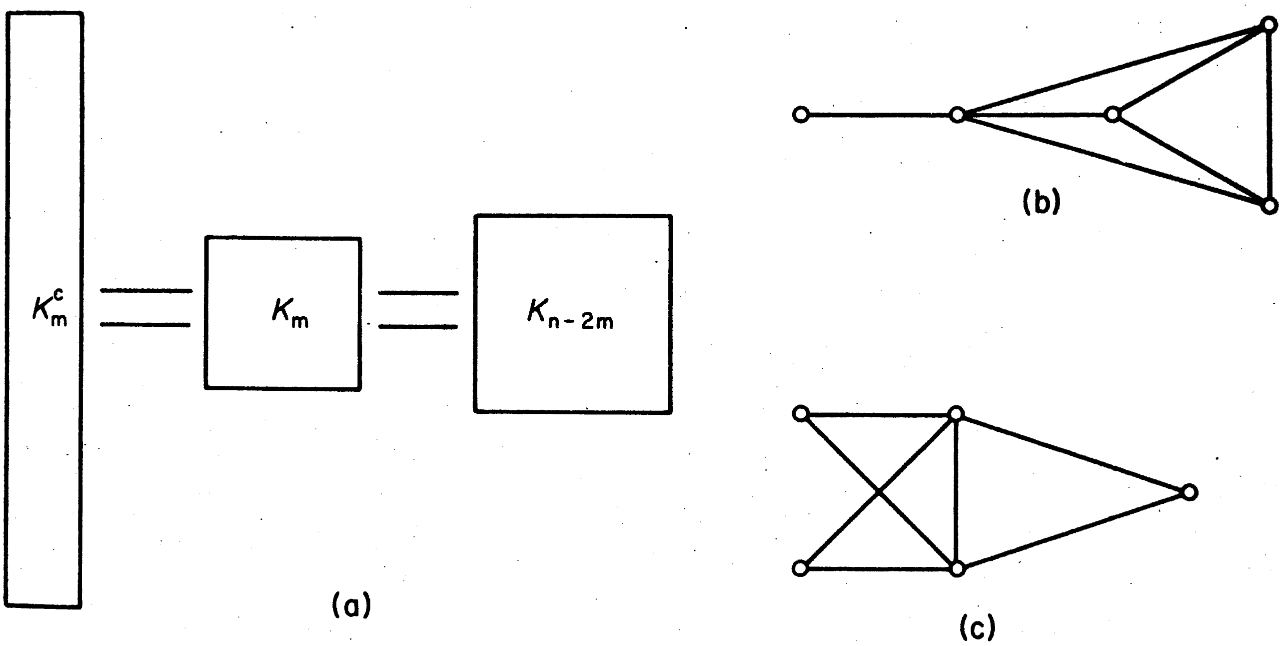


Figure 4.9. (*a*) $C_{m,n}$; (*b*) $C_{1,5}$; (*c*) $C_{2,5}$

From theorem 4.6 we can deduce a result due to Ore (1961) and Bondy (1972).

Corollary 4.6 If G is a simple graph with $\nu \geq 3$ and $\varepsilon > \binom{\nu-1}{2}+1$, then G is hamiltonian. Moreover, the only nonhamiltonian simple graphs with ν vertices and $\binom{\nu-1}{2}+1$ edges are $C_{1,\nu}$ and, for $\nu=5$, $C_{2,5}$.

Proof Let G be a nonhamiltonian simple graph with $\nu \geq 3$. By theorem 4.6, G is degree-majorised by $C_{m,\nu}$ for some positive integer $m<\nu/2$. Therefore, by theorem 1.1,

$$\varepsilon(G) \leq \varepsilon(C_{m,\nu}) \tag{4.9}$$

$$= \tfrac{1}{2}(m^2+(\nu-2m)(\nu-m-1)+m(\nu-1))$$

$$= \binom{\nu-1}{2}+1-\tfrac{1}{2}(m-1)(m-2)-(m-1)(\nu-2m-1)$$

$$\leq \binom{\nu-1}{2}+1 \tag{4.10}$$

Furthermore, equality can only hold in (4.9) if G has the same degree sequence as $C_{m,\nu}$; and equality can only hold in (4.10) if either $m=2$ and $\nu=5$, or $m=1$. Hence $\varepsilon(G)$ can equal $\binom{\nu-1}{2}+1$ only if G has the same degree sequence as $C_{1,\nu}$ or $C_{2,5}$, which is easily seen to imply that $G \cong C_{1,\nu}$ or $G \cong C_{2,5}$ □

Exercises

4.2.1 Show that if either
(*a*) G is not 2-connected, or
(*b*) G is bipartite with bipartition (X, Y) where $|X| \neq |Y|$,
then G is nonhamiltonian.

4.2.2 A mouse eats his way through a $3\times3\times3$ cube of cheese by tunnelling through all of the 27 $1\times1\times1$ subcubes. If he starts at one corner and always moves on to an uneaten subcube, can he finish at the centre of the cube?

4.2.3 Show that if G has a Hamilton path then, for every proper subset S of V, $\omega(G-S) \leq |S|+1$.

4.2.4* Let G be a nontrivial simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$. Show that if there is no value of m less than $(\nu+1)/2$ for which $d_m < m$ and $d_{\nu-m+1} < \nu - m$, then G has a Hamilton path. (V. Chvátal)

4.2.5 (*a*) Let G be a simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$ and let G^c have degree sequence $(d'_1, d'_2, \ldots, d'_\nu)$ where $d_1 \le d_2 \le \ldots \le d_\nu$ and $d'_1 \le d'_2 \le \ldots \le d'_\nu$. Show that if $d_m \ge d'_m$ for all $m \le \nu/2$, then G has a Hamilton path.

(*b*) Deduce that if G is self-complementary, then G has a Hamilton path. (C. R. J. Clapham)

4.2.6* Let G be a simple bipartite graph with bipartition (X, Y), where $|X| = |Y| \ge 2$, and let G have degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \le d_2 \le \ldots \le d_\nu$. Show that if there is no value of m less than or equal to $\nu/4$ for which $d_m \le m$ and $d_{\nu/2} \le \nu/2 - m$, then G is hamiltonian. (V. Chvátal)

4.2.7 Prove corollary 4.6 directly from corollary 4.4.

4.2.8 Show that if G is simple with $\nu \ge 6\delta$ and $\varepsilon > \binom{\nu-\delta}{2} + \delta^2$, then G is hamiltonian. (P. Erdös)

4.2.9* Show that if G is a connected graph with $\nu > 2\delta$, then G has a path of length at least 2δ. (G. A. Dirac)
(Dirac, 1952 has also shown that if G is a 2-connected simple graph with $\nu \ge 2\delta$, then G has a cycle of length at least 2δ.)

4.2.10 Using the remark to exercise 4.2.9, show that every $2k$-regular simple graph on $4k+1$ vertices is hamiltonian $(k \ge 1)$.
(C. St. J. A. Nash-Williams)

4.2.11 G is *Hamilton-connected* if every two vertices of G are connected by a Hamilton path.

(*a*) Show that if G is Hamilton-connected and $\nu \ge 4$, then $\varepsilon \ge [\frac{1}{2}(3\nu+1)]$.

(*b*)* For $\nu \ge 4$, construct a Hamilton-connected graph G with $\varepsilon = [\frac{1}{2}(3\nu+1)]$. (J. W. Moon)

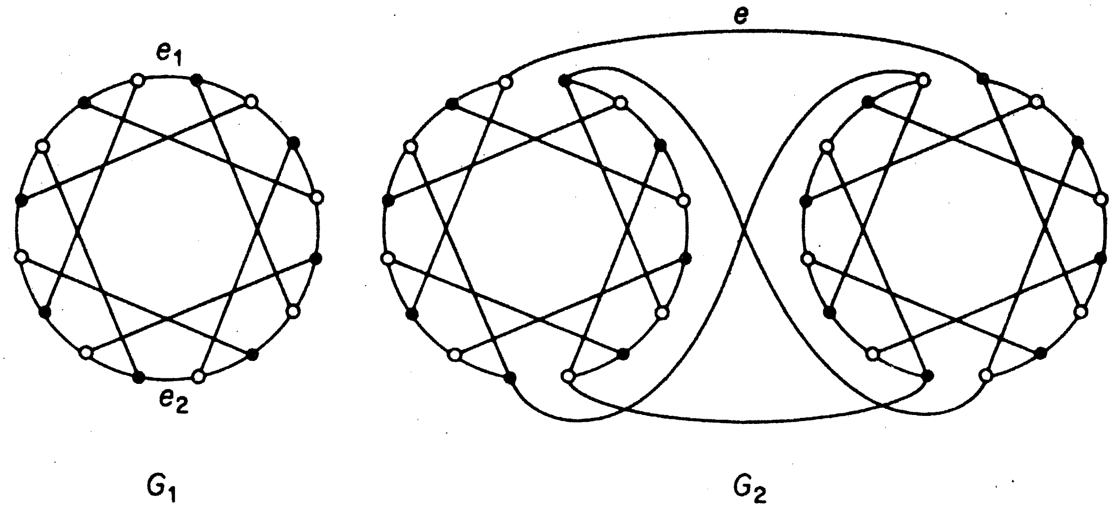
4.2.12 G is *hypohamiltonian* if G is not hamiltonian but $G - v$ is hamiltonian for every $v \in V$. Show that the Petersen graph (figure 4.4) is hypohamiltonian.
(Herz, Duby and Vigué, 1967 have shown that it is, in fact, the smallest such graph.)

4.2.13* G is *hypotraceable* if G has no Hamilton path but $G - v$ has a Hamilton path for every $v \in V$. Show that the Thomassen graph (p. 240) is hypotraceable.

4.2.14 (*a*) Show that there is no Hamilton cycle in the graph G_1 below which contains exactly one of the edges e_1 and e_2.

(*b*) Using (*a*), show that every Hamilton cycle in G_2 includes the edge e.

(*c*) Deduce that the Horton graph (p. 240) is nonhamiltonian.



4.2.15 Describe a good algorithm for

(*a*) constructing the closure of a graph;

(*b*) finding a Hamilton cycle if the closure is complete.

APPLICATIONS

4.3 THE CHINESE POSTMAN PROBLEM

In his job, a postman picks up mail at the post office, delivers it, and then returns to the post office. He must, of course, cover each street in his area at least once. Subject to this condition, he wishes to choose his route in such a way that he walks as little as possible. This problem is known as the *Chinese postman problem*, since it was first considered by a Chinese mathematician, Kuan (1962).

In a weighted graph, we define the *weight* of a tour $v_0e_1v_1 \dots e_nv_0$ to be $\sum_{i=1}^{n} w(e_i)$. Clearly, the Chinese postman problem is just that of finding a minimum-weight tour in a weighted connected graph with non-negative weights. We shall refer to such a tour as an *optimal tour.*

If G is eulerian, then any Euler tour of G is an optimal tour because an Euler tour is a tour that traverses each edge exactly once. The Chinese postman problem is easily solved in this case, since there exists a good algorithm for determining an Euler tour in an eulerian graph. The algorithm, due to Fleury (see Lucas, 1921), constructs an Euler tour by tracing out a trail, subject to the one condition that, at any stage, a cut edge of the untraced subgraph is taken only if there is no alternative.

Fleury's Algorithm

1. Choose an arbitrary vertex v_0, and set $W_0 = v_0$.
2. Suppose that the trail $W_i = v_0e_1v_1 \dots e_iv_i$ has been chosen.

Then choose an edge e_{i+1} from $E\backslash\{e_1, e_2, \ldots, e_i\}$ in such a way that

(i) e_{i+1} is incident with v_i;
(ii) unless there is no alternative, e_{i+1} is not a cut edge of

$$G_i = G - \{e_1, e_2, \ldots, e_i\}$$

3. Stop when step 2 can no longer be implemented.

By its definition, Fleury's algorithm constructs a trail in G.

Theorem 4.7 If G is eulerian, then any trail in G constructed by Fleury's algorithm is an Euler tour of G.

Proof Let G be eulerian, and let $W_n = v_0e_1v_1 \ldots e_nv_n$ be a trail in G constructed by Fleury's algorithm. Clearly, the terminus v_n must be of degree zero in G_n. It follows that $v_n = v_0$; in other words, W_n is a closed trail.

Suppose, now, that W_n is not an Euler tour of G, and let S be the set of vertices of positive degree in G_n. Then S is nonempty and $v_n \in \bar{S}$, where $\bar{S} = V\backslash S$. Let m be the largest integer such that $v_m \in S$ and $v_{m+1} \in \bar{S}$. Since W_n terminates in $\bar{S}$, e_{m+1} is the only edge of $[S, \bar{S}]$ in G_m, and hence is a cut edge of G_m (see figure 4.10).

Let e be any other edge of G_m incident with v_m. It follows (step 2) that e must also be a cut edge of G_m, and hence of $G_m[S]$. But since $G_m[S] = G_n[S]$, every vertex in $G_m[S]$ is of even degree. However, this implies (exercise 2.2.6*a*) that $G_m[S]$ has no cut edge, a contradiction □

The proof that Fleury's algorithm is a good algorithm is left as an exercise (exercise 4.3.2).

If G is not eulerian, then any tour in G and, in particular, an optimal tour in G, traverses some edges more than once. For example, in the graph of figure 4.11*a* *xuywvzwyxuwvxzyx* is an optimal tour (exercise 4.3.1). Notice that the four edges *ux*, *xy*, *yw* and *wv* are traversed twice by this tour.

It is convenient, at this stage, to introduce the operation of duplication of an edge. An edge e is said to be *duplicated* when its ends are joined by a

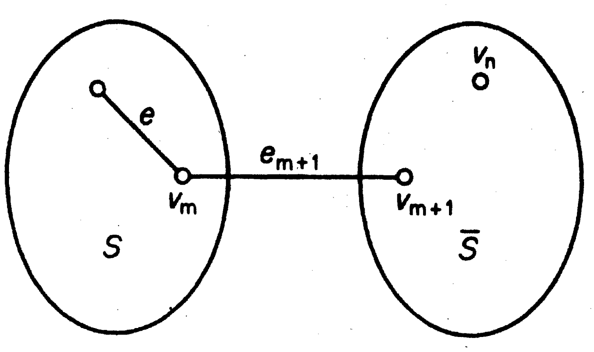
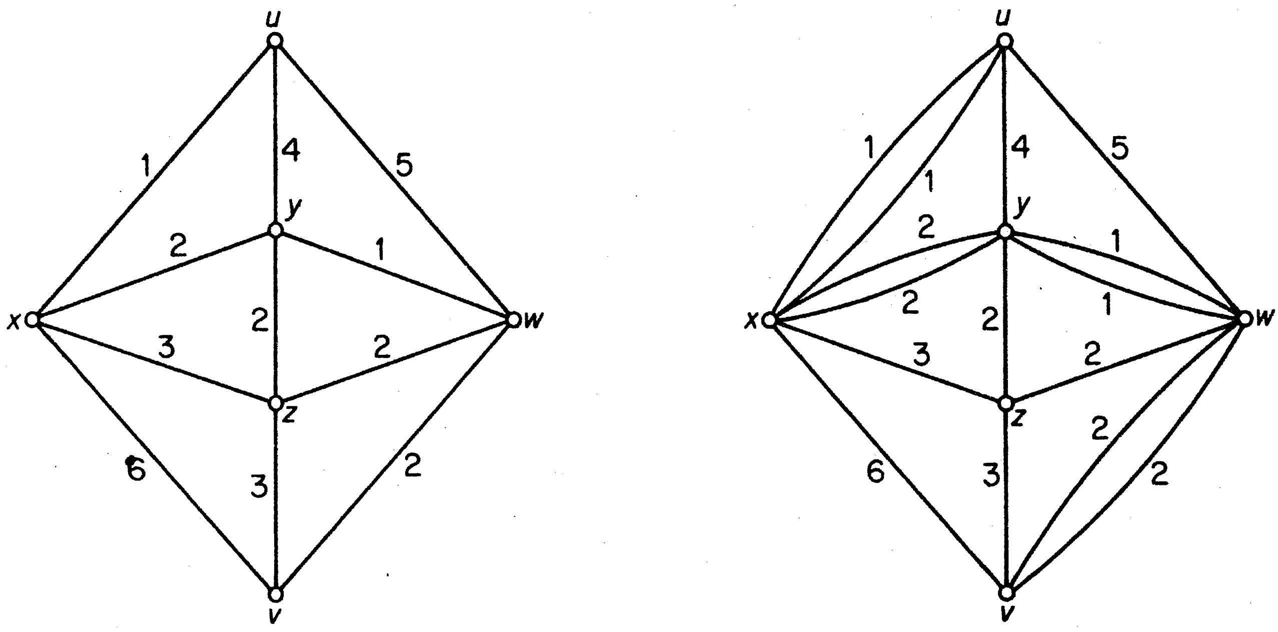


Figure 4.10



(a) (b)

Figure 4.11

new edge of weight $w(e)$. By duplicating the edges ux, xy, yw and wv in the graph of figure 4.11a, we obtain the graph shown in figure 4.11b.

We may now rephrase the Chinese postman problem as follows: given a weighted graph G with non-negative weights,

(i) find, by duplicating edges, an eulerian weighted supergraph G^* of G such that $\sum_{e \in E(G^*)\setminus E(G)} w(e)$ is as small as possible;

(ii) find an Euler tour in G^*.

That this is equivalent to the Chinese postman problem follows from the observation that a tour of G in which edge e is traversed $m(e)$ times corresponds to an Euler tour in the graph obtained from G by duplicating e $m(e)-1$ times, and vice versa.

We have already presented a good algorithm for solving (ii), namely Fleury's algorithm. A good algorithm for solving (i) has been given by Edmonds and Johnson (1973). Unfortunately, it is too involved to be presented here. However, we shall consider one special case which affords an easy solution. This is the case where G has exactly two vertices of odd degree.

Suppose that G has exactly two vertices u and v of odd degree; let G^* be an eulerian spanning supergraph of G obtained by duplicating edges, and write E^* for $E(G^*)$. Clearly the subgraph $G^*[E^*\setminus E]$ of G^* (induced by the edges of G^* that are not in G) also has only the two vertices u and v of odd degree. It follows from corollary 1.1 that u and v are in the same component of $G^*[E^*\setminus E]$ and hence that they are connected by a (u, v)-path P^*.

Clearly

$$\sum_{e \in E^* \setminus E} w(e) \geq w(P^*) \geq w(P)$$

where P is a minimum-weight (u, v)-path in G. Thus $\sum_{e \in E^* \setminus E} w(e)$ is a minimum when G^* is obtained from G by duplicating each of the edges on a minimum-weight (u, v)-path. A good algorithm for finding such a path was given in section 1.8.

Exercises

4.3.1 Show that *xuywvzwyxuwvxzyx* is an optimal tour in the weighted graph of figure 4.11*a*.

4.3.2 Draw a flow diagram summarising Fleury's algorithm, and show that it is a good algorithm.

4.4 THE TRAVELLING SALESMAN PROBLEM

A travelling salesman wishes to visit a number of towns and then return to his starting point. Given the travelling times between towns, how should he plan his itinerary so that he visits each town exactly once and travels in all for as short a time as possible? This is known as the *travelling salesman problem*. In graphical terms, the aim is to find a minimum-weight Hamilton cycle in a weighted complete graph. We shall call such a cycle an *optimal cycle*. In contrast with the shortest path problem and the connector problem, no efficient algorithm for solving the travelling salesman problem is known. It is therefore desirable to have a method for obtaining a reasonably good (but not necessarily optimal) solution. We shall show how some of our previous theory can be employed to this end.

One possible approach is to first find a Hamilton cycle C, and then search for another of smaller weight by suitably modifying C. Perhaps the simplest such modification is as follows.

Let $C = v_1 v_2 \ldots v_\nu v_1$. Then, for all i and j such that $1 < i+1 < j < \nu$, we can obtain a new Hamilton cycle

$$C_{ij} = v_1 v_2 \ldots v_i v_j v_{j-1} \ldots v_{i+1} v_{j+1} v_{j+2} \ldots v_\nu v_1$$

by deleting the edges $v_i v_{i+1}$ and $v_j v_{j+1}$ and adding the edges $v_i v_j$ and $v_{i+1} v_{j+1}$, as shown in figure 4.12.

If, for some i and j

$$w(v_i v_j) + w(v_{i+1} v_{j+1}) < w(v_i v_{i+1}) + w(v_j v_{j+1})$$

the cycle C_{ij} will be an improvement on C.

After performing a sequence of the above modifications, one is left with a cycle that can be improved no more by these methods. This final cycle will

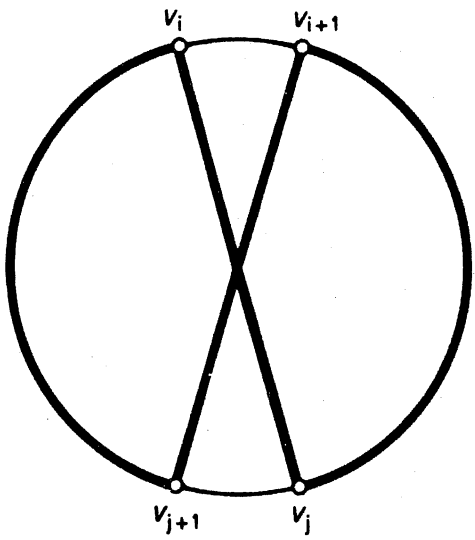


Figure 4.12

almost certainly not be optimal, but it is a reasonable assumption that it will often be fairly good; for greater accuracy, the procedure can be repeated several times, starting with a different cycle each time.

As an example, consider the weighted graph shown in figure 4.13; it is the same graph as was used in our illustration of Kruskal's algorithm in section 2.5.

Starting with the cycle L MC NY Pa Pe T L, we can apply a sequence of three modifications, as illustrated in figure 4.14, and end up with the cycle L NY MC T Pe Pa L of weight 192.

An indication of how good our solution is can sometimes be obtained by applying Kruskal's algorithm. Suppose that C is an optimal cycle in G. Then, for any vertex v, $C-v$ is a Hamilton path in $G-v$, and is therefore a

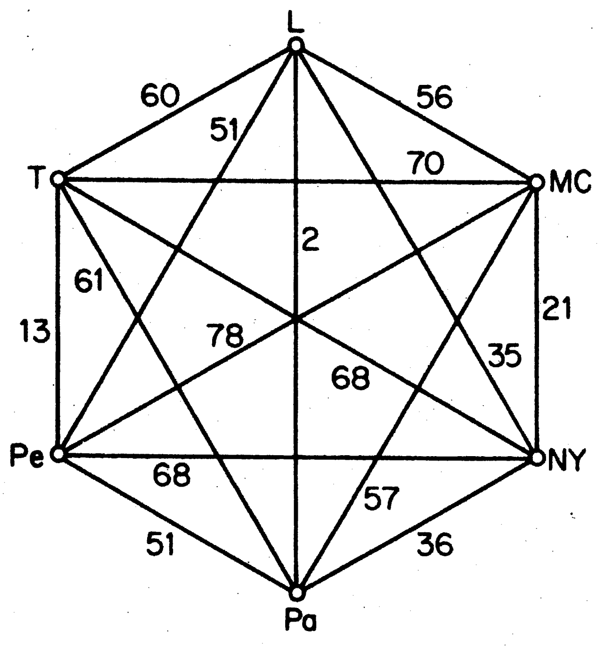


Figure 4.13

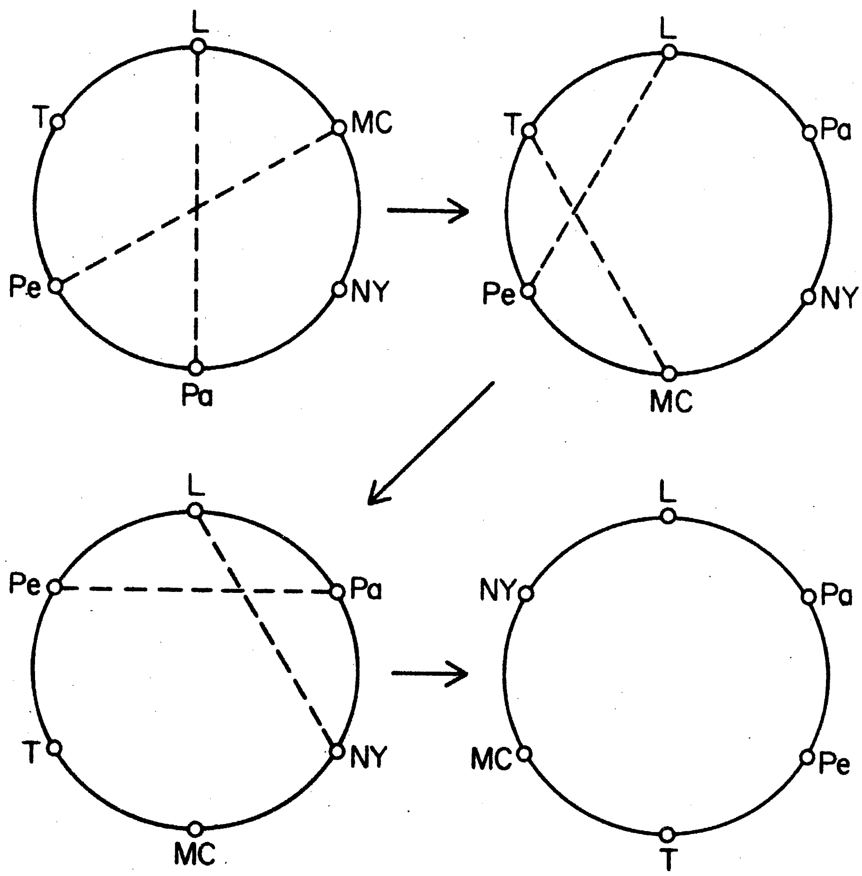


Figure 4.14

spanning tree of $G-v$. It follows that if T is an optimal tree in $G-v$, and if e and f are two edges incident with v such that $w(e)+w(f)$ is as small as possible, then $w(T)+w(e)+w(f)$ will be a lower bound on $w(C)$. In our example, taking NY as the vertex v, we find (see figure 4.15) that

$$w(T) = 122 \qquad w(e) = 21 \quad \text{and} \quad w(f) = 35$$

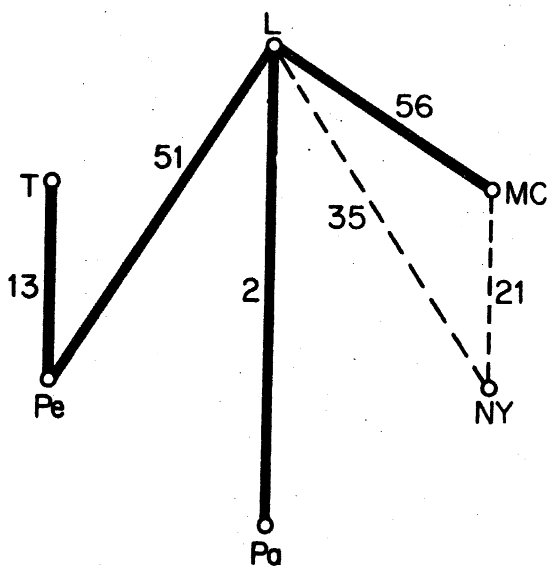


Figure 4.15

We can therefore conclude that the weight $w(C)$ of an optimal cycle in the graph of figure 4.13 satisfies

$$178 \leq w(C) \leq 192$$

The methods described here have been further developed by Lin (1965) and Held and Karp (1970; 1971). In particular, Lin has found that the cycle modification procedure can be made more efficient by replacing three edges at a time rather than just two; somewhat surprisingly, however, it is not advantageous to extend this same idea further. For a survey of the travelling salesman problem, see Bellmore and Nemhauser (1968).

Exercise

4.4.1* Let G be a weighted complete graph in which the weights satisfy the triangle inequality: $w(xy)+w(yz) \geq w(xz)$ for all x, y, $z \in V$. Show that an optimal cycle in G has weight at most $2w(T)$, where T is an optimal tree in G.

(D. J. Rosencrantz, R. E. Stearns, P. M. Lewis)

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5 Matchings

5.1 MATCHINGS

A subset M of E is called a *matching* in G if its elements are links and no two are adjacent in G; the two ends of an edge in M are said to be *matched under* M. A matching M *saturates* a vertex v, and v is said to be M-*saturated*, if some edge of M is incident with v; otherwise, v is M-*unsaturated*. If every vertex of G is M-saturated, the matching M is *perfect*. M is a *maximum matching* if G has no matching M' with $|M'|>|M|$; clearly, every perfect matching is maximum. Maximum and perfect matchings in graphs are indicated in figure 5.1.

Let M be a matching in G. An M-*alternating path* in G is a path whose edges are alternately in $E\backslash M$ and M. For example, the path $v_5v_8v_1v_7v_6$ in the graph of figure 5.1a is an M-alternating path. An M-*augmenting path* is an M-alternating path whose origin and terminus are M-unsaturated.

Theorem 5.1 (Berge, 1957) A matching M in G is a maximum matching if and only if G contains no M-augmenting path.

Proof Let M be a matching in G, and suppose that G contains an M-augmenting path $v_0v_1 \ldots v_{2m+1}$. Define $M' \subseteq E$ by

$$M' = (M\backslash\{v_1v_2, v_3v_4, \ldots, v_{2m-1}v_{2m}\}) \cup \{v_0v_1, v_2v_3, \ldots, v_{2m}v_{2m+1}\}$$

Then M' is a matching in G, and $|M'| = |M|+1$. Thus M is not a maximum matching.

Conversely, suppose that M is not a maximum matching, and let M' be a maximum matching in G. Then

$$|M'|>|M| \tag{5.1}$$

Set $H = G[M \Delta M']$, where $M \Delta M'$ denotes the symmetric difference of M and M' (see figure 5.2).

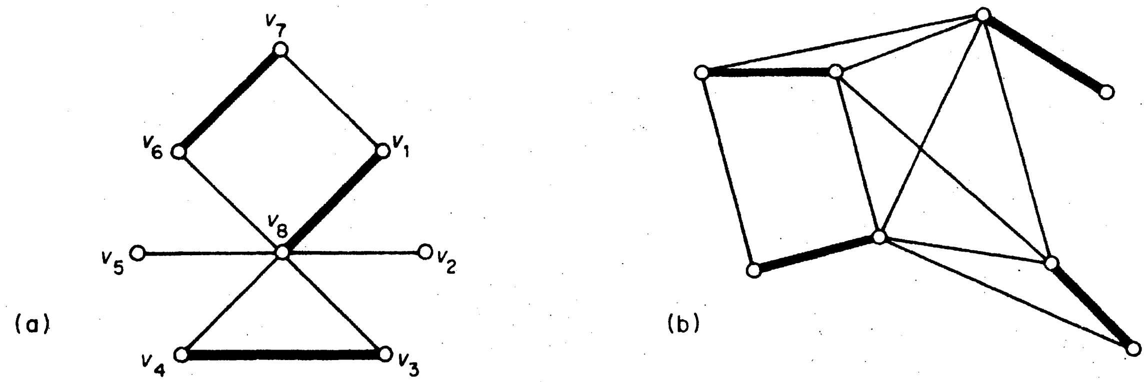


Figure 5.1. (*a*) A maximum matching; (*b*) a perfect matching

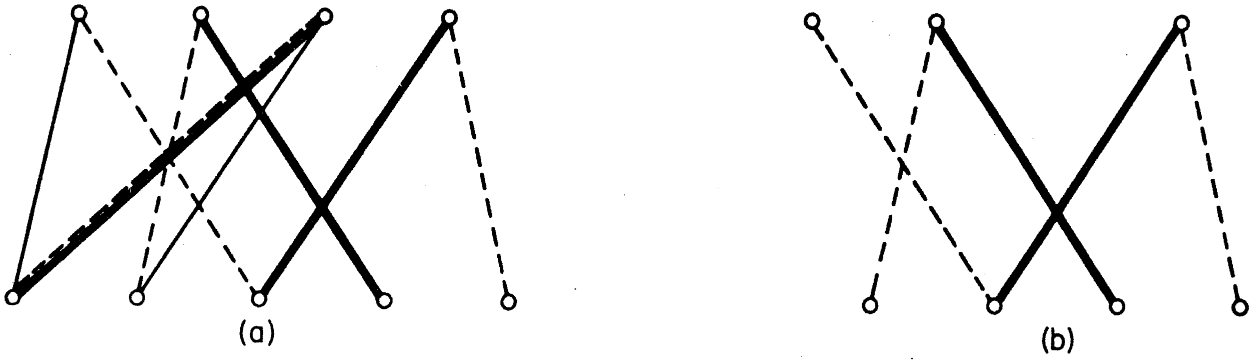


Figure 5.2. (*a*) G, with M heavy and M' broken; (*b*) $G[M \Delta M']$

Each vertex of H has degree either one or two in H, since it can be incident with at most one edge of M and one edge of M'. Thus each component of H is either an even cycle with edges alternately in M and M', or else a path with edges alternately in M and M'. By (5.1), H contains more edges of M' than of M, and therefore some path component P of H must start and end with edges of M'. The origin and terminus of P, being M'-saturated in H, are M-unsaturated in G. Thus P is an M-augmenting path in G □

Exercises

5.1.1 (*a*) Show that every k-cube has a perfect matching ($k \geq 2$).
(*b*) Find the number of different perfect matchings in K_{2n} and $K_{n,n}$.

5.1.2 Show that a tree has at most one perfect matching.

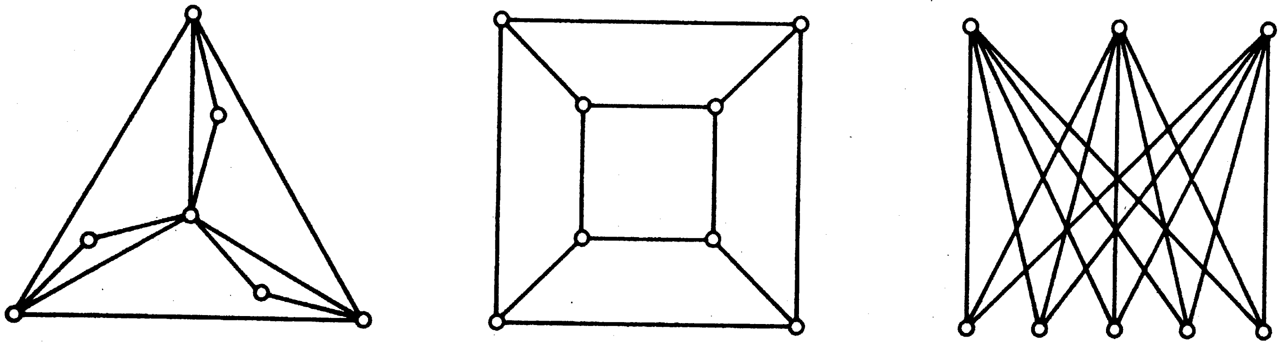
5.1.3 For each $k > 1$, find an example of a k-regular simple graph that has no perfect matching.

5.1.4 Two people play a game on a graph G by alternately selecting distinct vertices $v_0, v_1, v_2, \ldots$ such that, for $i > 0$, v_i is adjacent to v_{i-1}. The last player able to select a vertex wins. Show that the first player has a winning strategy if and only if G has no perfect matching.

5.1.5 A *k-factor* of G is a k-regular spanning subgraph of G, and G is *k-factorable* if there are edge-disjoint k-factors $H_1, H_2, \ldots, H_n$ such that $G = H_1 \cup H_2 \cup \ldots \cup H_n$.

(*a*)* Show that
(i) $K_{n,n}$ and K_{2n} are 1-factorable;
(ii) the Petersen graph is not 1-factorable.

(*b*) Which of the following graphs have 2-factors?



(c) Using Dirac's theorem (4.3), show that if G is simple, with ν even and $\delta \geq (\nu/2)+1$, then G has a 3-factor.

5.1.6* Show that K_{2n+1} can be expressed as the union of n connected 2-factors ($n \geq 1$).

5.2 MATCHINGS AND COVERINGS IN BIPARTITE GRAPHS

For any set S of vertices in G, we define the *neighbour set* of S in G to be the set of all vertices adjacent to vertices in S; this set is denoted by $N_G(S)$. Suppose, now, that G is a bipartite graph with bipartition (X, Y). In many applications one wishes to find a matching of G that saturates every vertex in X; an example is the personnel assignment problem, to be discussed in section 5.4. Necessary and sufficient conditions for the existence of such a matching were first given by Hall (1935).

Theorem 5.2 Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if

$$|N(S)| \geq |S| \quad \text{for all} \quad S \subseteq X \tag{5.2}$$

Proof Suppose that G contains a matching M which saturates every vertex in X, and let S be a subset of X. Since the vertices in S are matched under M with distinct vertices in $N(S)$, we clearly have $|N(S)| \geq |S|$.

Conversely, suppose that G is a bipartite graph satisfying (5.2), but that G contains no matching saturating all the vertices in X. We shall obtain a contradiction. Let M^* be a maximum matching in G. By our supposition, M^* does not saturate all vertices in X. Let u be an M^*-unsaturated vertex in X, and let Z denote the set of all vertices connected to u by M^*-alternating paths. Since M^* is a maximum matching, it follows from theorem 5.1 that u is the only M^*-unsaturated vertex in Z. Set $S = Z \cap X$ and $T = Z \cap Y$ (see figure 5.3).

Clearly, the vertices in $S \backslash \{u\}$ are matched under M^* with the vertices in T. Therefore

$$|T| = |S| - 1 \tag{5.3}$$

and $N(S) \supseteq T$. In fact, we have

$$N(S) = T \tag{5.4}$$

since every vertex in $N(S)$ is connected to u by an M^*-alternating path. But

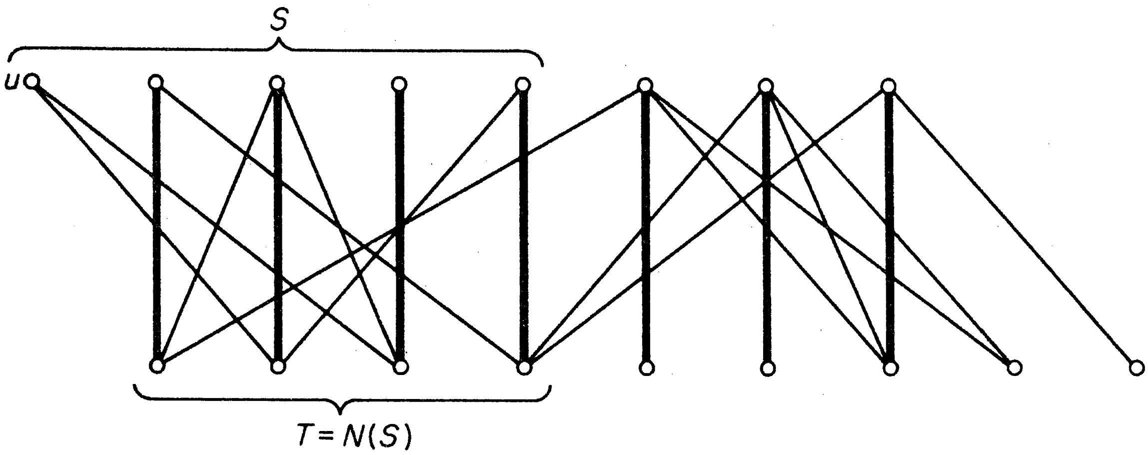


Figure 5.3

(5.3) and (5.4) imply that

$$|N(S)| = |S| - 1 < |S|$$

contradicting assumption (5.2) □

The above proof provides the basis of a good algorithm for finding a maximum matching in a bipartite graph. This algorithm will be presented in section 5.4.

Corollary 5.2 If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.

Proof Let G be a k-regular bipartite graph with bipartition (X, Y). Since G is k-regular, $k\,|X| = |E| = k\,|Y|$ and so, since $k > 0$, $|X| = |Y|$. Now let S be a subset of X and denote by E_1 and E_2 the sets of edges incident with vertices in S and $N(S)$, respectively. By definition of $N(S)$, $E_1 \subseteq E_2$ and therefore

$$k\,|N(S)| = |E_2| \geq |E_1| = k\,|S|$$

It follows that $|N(S)| \geq |S|$ and hence, by theorem 5.2, that G has a matching M saturating every vertex in X. Since $|X| = |Y|$, M is a perfect matching □

Corollary 5.2 is sometimes known as the *marriage theorem*, since it can be more colourfully restated as follows: if every girl in a village knows exactly k boys, and every boy knows exactly k girls, then each girl can marry a boy she knows, and each boy can marry a girl he knows.

A *covering* of a graph G is a subset K of V such that every edge of G has at least one end in K. A covering K is a *minimum covering* if G has no covering K' with $|K'| < |K|$ (see figure 5.4).

If K is a covering of G, and M is a matching of G, then K contains at

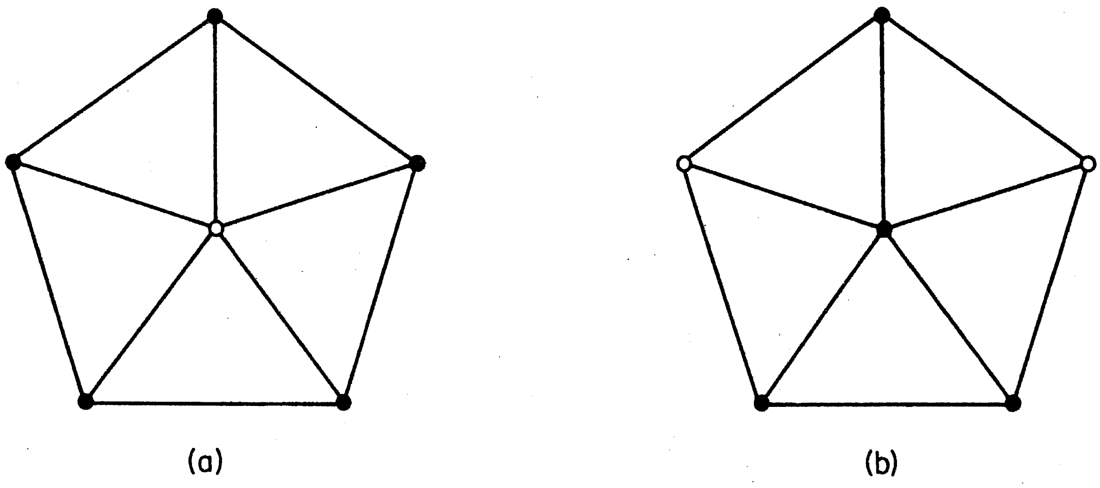


Figure 5.4. (*a*) A covering; (*b*) a minimum covering

least one end of each of the edges in M. Thus, for any matching M and any covering K, $|M| \leq |K|$. Indeed, if M^* is a maximum matching and $\tilde{K}$ is a minimum covering, then

$$|M^*| \leq |\tilde{K}| \tag{5.5}$$

In general, equality does not hold in (5.5) (see, for example, figure 5.4). However, if G is bipartite we do have $|M^*| = |\tilde{K}|$. This result, due to König (1931), is closely related to Hall's theorem. Before presenting its proof, we make a simple, but important, observation.

Lemma 5.3 Let M be a matching and K be a covering such that $|M| = |K|$. Then M is a maximum matching and K is a minimum covering.

Proof If M^* is a maximum matching and $\tilde{K}$ is a minimum covering then, by (5.5),

$$|M| \leq |M^*| \leq |\tilde{K}| \leq |K|$$

Since $|M| = |K|$, it follows that $|M| = |M^*|$ and $|K| = |\tilde{K}|$ □

Theorem 5.3 In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

Proof Let G be a bipartite graph with bipartition (X, Y), and let M^* be a maximum matching of G. Denote by U the set of M^*-unsaturated vertices in X, and by Z the set of all vertices connected by M^*-alternating paths to vertices of U. Set $S = Z \cap X$ and $T = Z \cap Y$. Then, as in the proof of theorem 5.2, we have that every vertex in T is M^*-saturated and $N(S) = T$. Define $\tilde{K} = (X \backslash S) \cup T$ (see figure 5.5). Every edge of G must have at least one of its ends in $\tilde{K}$. For, otherwise, there would be an edge with one end in

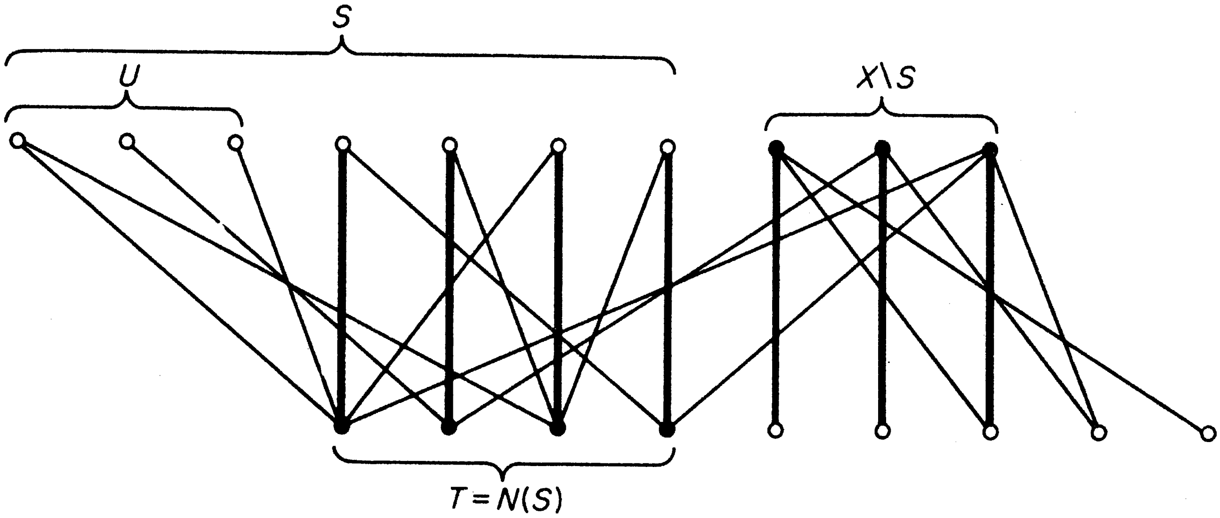


Figure 5.5

S and one end in $Y\backslash T$, contradicting $N(S)=T$. Thus $\tilde{K}$ is a covering of G and clearly

$$|M^*|=|\tilde{K}|$$

By lemma 5.3, $\tilde{K}$ is a minimum covering, and the theorem follows □

Exercises

5.2.1 Show that it is impossible, using 1×2 rectangles, to exactly cover an 8×8 square from which two opposite 1×1 corner squares have been removed.

5.2.2 (*a*) Show that a bipartite graph G has a perfect matching if and only if $|N(S)|\geq|S|$ for all $S\subseteq V$.
(*b*) Give an example to show that the above statement does not remain valid if the condition that G be bipartite is dropped.

5.2.3 For $k>0$, show that
(*a*) every k-regular bipartite graph is 1-factorable;
(*b*)* every $2k$-regular graph is 2-factorable. (J. Petersen)

5.2.4 Let $A_1, A_2, \ldots, A_m$ be subsets of a set S. A *system of distinct representatives* for the family $(A_1, A_2, \ldots, A_m)$ is a subset $\{a_1, a_2, \ldots, a_m\}$ of S such that $a_i\in A_i$, $1\leq i\leq m$, and $a_i\neq a_j$ for $i\neq j$. Show that $(A_1, A_2, \ldots, A_m)$ has a system of distinct representatives if and only if $\left|\bigcup_{i\in J} A_i\right|\geq|J|$ for all subsets J of $\{1, 2, \ldots, m\}$. (P. Hall)

5.2.5 A *line* of a matrix is a row or a column of the matrix. Show that the minimum number of lines containing all the 1's of a (0, 1)-matrix is equal to the maximum number of 1's, no two of which are in the same line.

5.2.6 (*a*) Prove the following generalisation of Hall's theorem (5.2): if G is a bipartite graph with bipartition (X, Y), the number of edges in a maximum matching of G is

$$|X| - \max_{S \subseteq X} \{|S| - |N(S)|\}$$

(D. König, O. Ore)

(*b*) Deduce that if G is simple with $|X| = |Y| = n$ and $\varepsilon > (k-1)n$, then G has a matching of cardinality k.

5.2.7 Deduce Hall's theorem (5.2) from König's theorem (5.3).

5.2.8* A non-negative real matrix $\mathbf{Q}$ is *doubly stochastic* if the sum of the entries in each row of $\mathbf{Q}$ is 1 and the sum of the entries in each column of $\mathbf{Q}$ is 1. A *permutation matrix* is a (0, 1)-matrix which has exactly one 1 in each row and each column. (Thus every permutation matrix is doubly stochastic.) Show that

(*a*) every doubly stochastic matrix is necessarily square;

(*b*) every doubly stochastic matrix $\mathbf{Q}$ can be expressed as a convex linear combination of permutation matrices; that is

$$\mathbf{Q} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2 + \ldots + c_k\mathbf{P}_k$$

where each $\mathbf{P}_i$ is a permutation matrix, each c_i is a non-negative real number, and $\sum_1^k c_i = 1$. (G. Birkhoff, J. von Neumann)

5.2.9 Let H be a finite group and let K be a subgroup of H. Show that there exist elements $h_1, h_2, \ldots, h_n \in H$ such that $h_1K, h_2K, \ldots, h_nK$ are the left cosets of K and $Kh_1, Kh_2, \ldots, Kh_n$ are the right cosets of K. (P. Hall)

5.3 PERFECT MATCHINGS

A necessary and sufficient condition for a graph to have a perfect matching was obtained by Tutte (1947). The proof given here is due to Lovász (1973).

A component of a graph is *odd* or *even* according as it has an odd or even number of vertices. We denote by $o(G)$ the number of odd components of G.

Theorem 5.4 G has a perfect matching if and only if

$$o(G-S) \leq |S| \quad \text{for all} \quad S \subset V \tag{5.6}$$

Proof It clearly suffices to prove the theorem for simple graphs.

Suppose first that G has a perfect matching M. Let S be a proper subset of V, and let $G_1, G_2, \ldots, G_n$ be the odd components of $G-S$. Because G_i is odd, some vertex u_i of G_i must be matched under M with a vertex v_i of S (see figure 5.6). Therefore, since $\{v_1, v_2, \ldots, v_n\} \subseteq S$

$$o(G-S) = n = |\{v_1, v_2, \ldots, v_n\}| \leq |S|$$

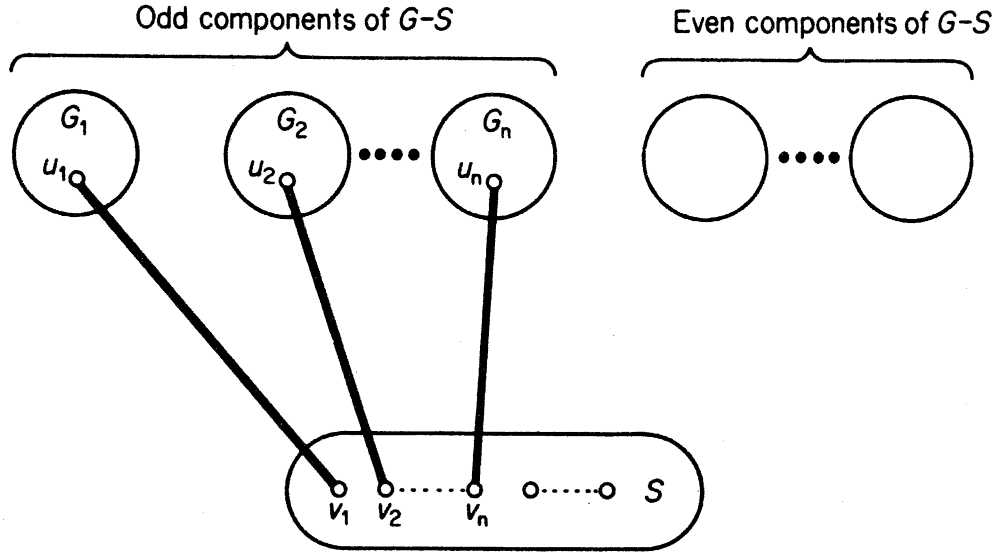


Figure 5.6

Conversely, suppose that G satisfies (5.6) but has no perfect matching. Then G is a spanning subgraph of a maximal graph G^* having no perfect matching. Since $G-S$ is a spanning subgraph of G^*-S we have $o(G^*-S) \leq o(G-S)$ and so, by (5.6),

$$o(G^*-S) \leq |S| \quad \text{for all} \quad S \subset V(G^*) \tag{5.7}$$

In particular, setting $S = \emptyset$, we see that $o(G^*) = 0$, and so $\nu(G^*)$ is even.

Denote by U the set of vertices of degree $\nu - 1$ in G^*. Since G^* clearly has a perfect matching if $U = V$, we may assume that $U \neq V$. We shall show that $G^* - U$ is a disjoint union of complete graphs. Suppose, to the contrary, that some component of $G^* - U$ is not complete. Then, in this component, there are vertices x, y and z such that $xy \in E(G^*)$, $yz \in E(G^*)$ and $xz \notin E(G^*)$ (exercise 1.6.14). Moreover, since $y \notin U$, there is a vertex w in $G^* - U$ such that $yw \notin E(G^*)$. The situation is illustrated in figure 5.7.

Since G^* is a maximal graph containing no perfect matching, $G^* + e$ has a perfect matching for all $e \notin E(G^*)$. Let M_1 and M_2 be perfect matchings in $G^* + xz$ and $G^* + yw$, respectively, and denote by H the subgraph of

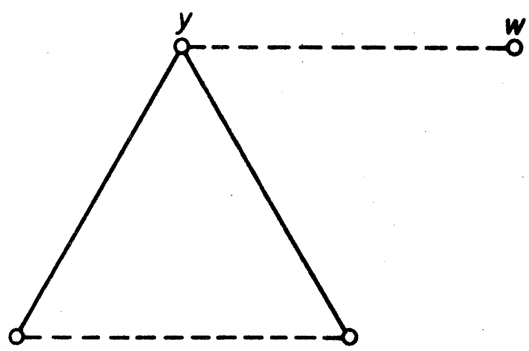


Figure 5.7

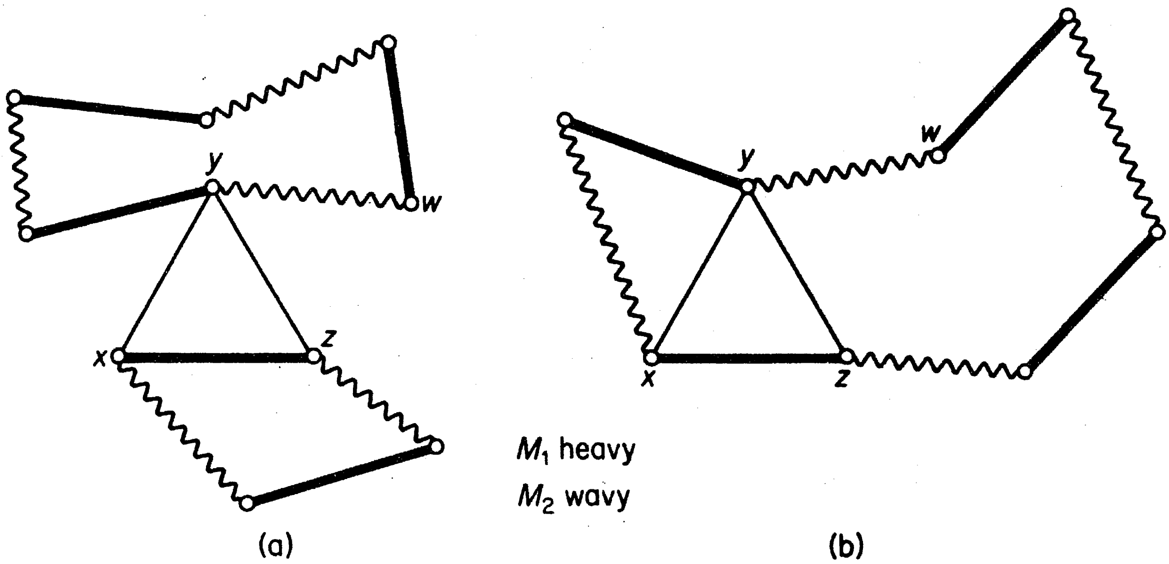


Figure 5.8

$G^* \cup \{xz, yw\}$ induced by $M_1 \Delta M_2$. Since each vertex of H has degree two, H is a disjoint union of cycles. Furthermore, all of these cycles are even, since edges of M_1 alternate with edges of M_2 around them. We distinguish two cases:

Case 1 xz and yw are in different components of H (figure 5.8*a*). Then, if yw is in the cycle C of H, the edges of M_1 in C, together with the edges of M_2 not in C, constitute a perfect matching in G^*, contradicting the definition of G^*.

Case 2 xz and yw are in the same component C of H. By symmetry of x and z, we may assume that the vertices x, y, w and z occur in that order on C (figure 5.8*b*). Then the edges of M_1 in the section $yw \ldots z$ of C, together with the edge yz and the edges of M_2 not in the section $yw \ldots z$ of C,

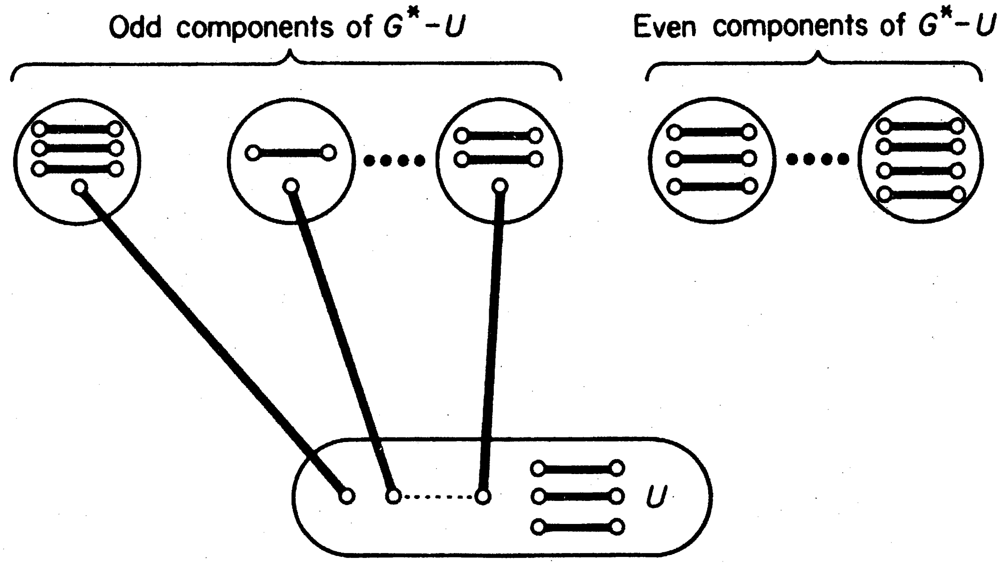


Figure 5.9

constitute a perfect matching in G^*, again contradicting the definition of G^*.

Since both case 1 and case 2 lead to contradictions, it follows that G^*-U is indeed a disjoint union of complete graphs.

Now, by (5.7), $o(G^*-U)\leq|U|$. Thus at most $|U|$ of the components of G^*-U are odd. But then G^* clearly has a perfect matching: one vertex in each odd component of G^*-U is matched with a vertex of U; the remaining vertices in U, and in components of G^*-U, are then matched as indicated in figure 5.9.

Since G^* was assumed to have no perfect matching we have obtained the desired contradiction. Thus G does indeed have a perfect matching □

The above theorem can also be proved with the aid of Hall's theorem (see Anderson, 1971).

From Tutte's theorem, we now deduce a result first obtained by Petersen (1891).

Corollary 5.4 Every 3-regular graph without cut edges has a perfect matching.

Proof Let G be a 3-regular graph without cut edges, and let S be a proper subset of V. Denote by $G_1, G_2, \ldots, G_n$ the odd components of $G-S$, and let m_i be the number of edges with one end in G_i and one end in S, $1\leq i\leq n$. Since G is 3-regular

$$\sum_{v\in V(G_i)} d(v)=3\nu(G_i) \quad \text{for} \quad 1\leq i\leq n \tag{5.8}$$

and

$$\sum_{v\in S} d(v)=3\,|S| \tag{5.9}$$

By (5.8), $m_i=\sum_{v\in V(G_i)} d(v)-2\varepsilon(G_i)$ is odd. Now $m_i\neq 1$ since G has no cut edge. Thus

$$m_i\geq 3 \quad \text{for} \quad 1\leq i\leq n \tag{5.10}$$

It follows from (5.10) and (5.9) that

$$o(G-S)=n\leq\frac{1}{3}\sum_{i=1}^{n} m_i\leq\frac{1}{3}\sum_{v\in S} d(v)=|S|$$

Therefore, by theorem 5.4, G has a perfect matching □

A 3-regular graph with cut edges need not have a perfect matching. For example, it follows from theorem 5.4 that the graph G of figure 5.10 has no perfect matching, since $o(G-v)=3$.

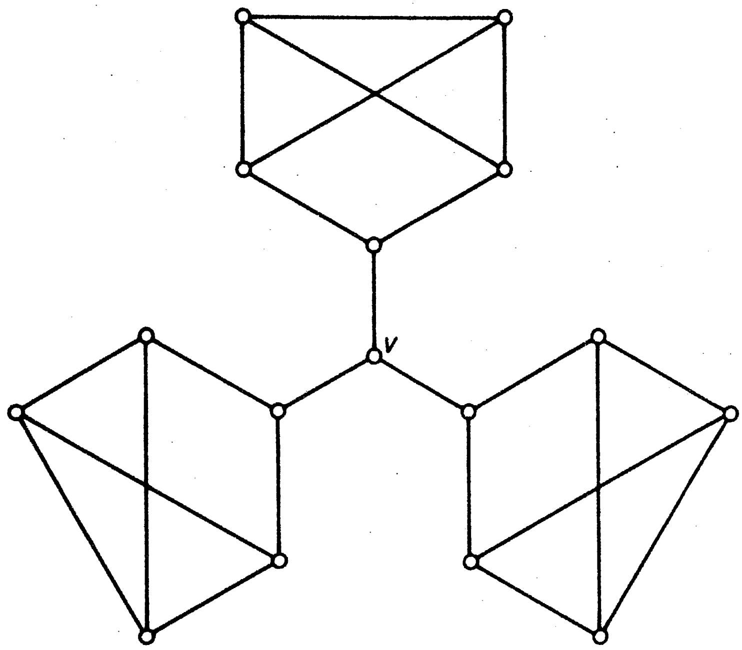


Figure 5.10

Exercises

5.3.1* Derive Hall's theorem (5.2) from Tutte's theorem (5.4).

5.3.2 Prove the following generalisation of corollary 5.4: if G is a $(k-1)$-edge-connected k-regular graph with ν even, then G has a perfect matching.

5.3.3 Show that a tree G has a perfect matching if and only if $o(G-v)=1$ for all $v \in V$. (V. Chungphaisan)

5.3.4* Prove the following generalisation of Tutte's theorem (5.4): the number of edges in a maximum matching of G is $\frac{1}{2}(\nu-d)$, where $d = \max_{S \subseteq V}\{o(G-S)-|S|\}$. (C. Berge)

5.3.5 (*a*) Using Tutte's theorem (5.4), characterise the maximal simple graphs which have no perfect matching.

(*b*) Let G be simple, with ν even and $\delta < \nu/2$. Show that if $\varepsilon > \binom{\delta}{2} + \binom{\nu-2\delta-1}{2} + \delta(\nu-\delta)$, then G has a perfect matching.

APPLICATIONS

5.4 THE PERSONNEL ASSIGNMENT PROBLEM

In a certain company, n workers $X_1, X_2, \ldots, X_n$ are available for n jobs $Y_1, Y_2, \ldots, Y_n$, each worker being qualified for one or more of these jobs. Can all the men be assigned, one man per job, to jobs for which they are qualified? This is the *personnel assignment problem*.

We construct a bipartite graph G with bipartition (X, Y), where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and x_i is joined to y_j if and only if worker X_i is qualified for job Y_j. The problem becomes one of determining whether or not G has a perfect matching. According to Hall's theorem (5.2), either G has such a matching or there is a subset S of X such that $|N(S)| < |S|$. In the sequel, we shall present an algorithm to solve the personnel assignment problem. Given any bipartite graph G with bipartition (X, Y), the algorithm either finds a matching of G that saturates every vertex in X or, failing this, finds a subset S of X such that $|N(S)| < |S|$.

The basic idea behind the algorithm is very simple. We start with an arbitrary matching M. If M saturates every vertex in X, then it is a matching of the required type. If not, we choose an M-unsaturated vertex u in X and systematically search for an M-augmenting path with origin u. Our method of search, to be described in detail below, finds such a path P if one exists; in this case $\hat{M} = M \,\Delta E(P)$ is a larger matching than M, and hence saturates more vertices in X. We then repeat the procedure with $\hat{M}$ instead of M. If such a path does not exist, the set Z of all vertices which are connected to u by M-alternating paths is found. Then (as in the proof of theorem 5.2) $S = Z \cap X$ satisfies $|N(S)| < |S|$.

Let M be a matching in G, and let u be an M-unsaturated vertex in X. A tree $H \subseteq G$ is called an *M-alternating tree rooted at u* if (i) $u \in V(H)$, and (ii) for every vertex v of H, the unique (u, v)-path in H is an M-alternating path. An M-alternating tree in a graph is shown in figure 5.11.

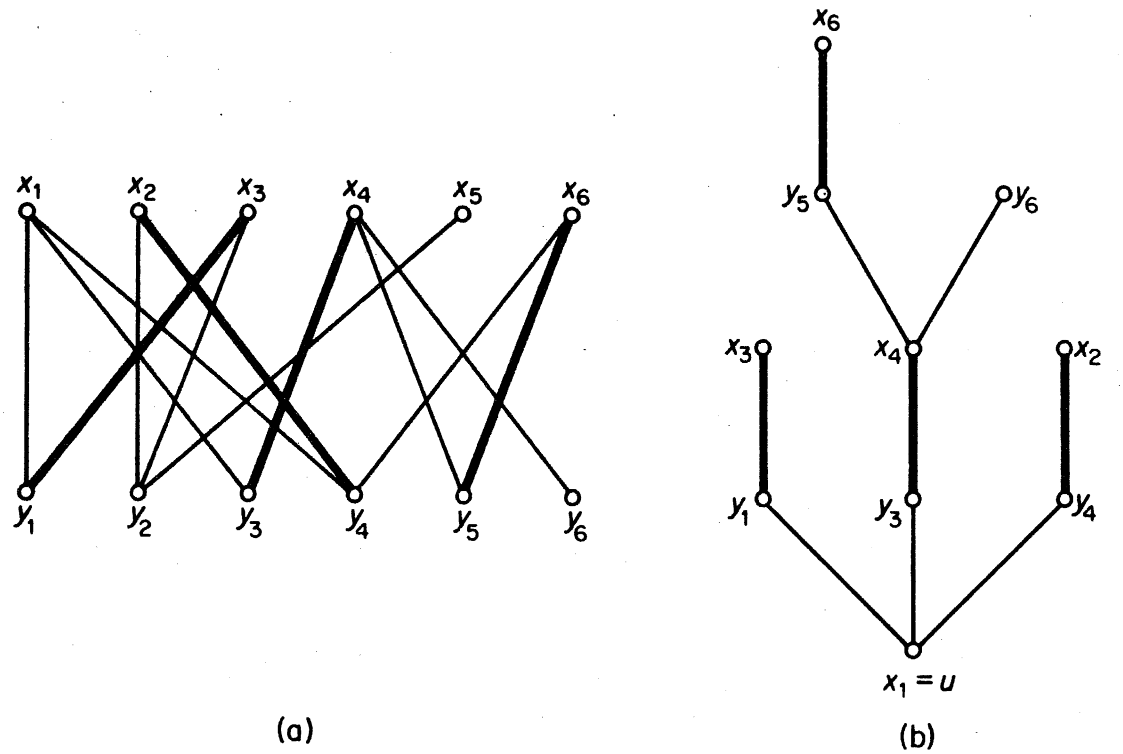


Figure 5.11. (*a*) A matching M in G; (*b*) an M-alternating tree in G

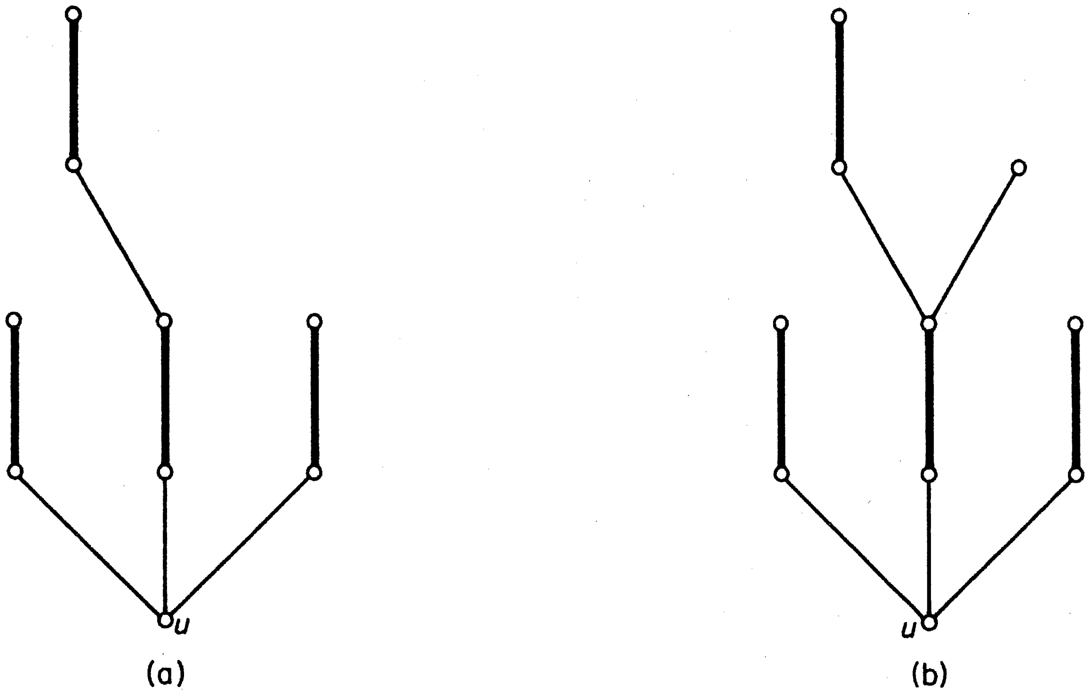


Figure 5.12. (*a*) Case (i); (*b*) case (ii)

The search for an M-augmenting path with origin u involves 'growing' an M-alternating tree H rooted at u. This procedure was first suggested by Edmonds (1965). Initially, H consists of just the single vertex u. It is then grown in such a way that, at any stage, either

(i) all vertices of H except u are M-saturated and matched under M (as in figure 5.12*a*), or
(ii) H contains an M-unsaturated vertex different from u (as in figure 5.12*b*).

If (i) is the case (as it is initially) then, setting $S = V(H) \cap X$ and $T = V(H) \cap Y$, we have $N(S) \supseteq T$; thus either $N(S) = T$ or $N(S) \supset T$.

(a) If $N(S) = T$ then, since the vertices in $S \backslash \{u\}$ are matched with the vertices in T, $|N(S)| = |S| - 1$, indicating that G has no matching saturating all vertices in X.
(b) If $N(S) \supset T$, there is a vertex y in $Y \backslash T$ adjacent to a vertex x in S. Since all vertices of H except u are matched under M, either $x = u$ or else x is matched with a vertex of H. Therefore $xy \notin M$. If y is M-saturated, with $yz \in M$, we grow H by adding the vertices y and z and the edges xy and yz. We are then back in case (i). If y is M-unsaturated, we grow H by adding the vertex y and the edge xy, resulting in case (ii). The (u, y)-path of H is then an M-augmenting path with origin u, as required.

Figure 5.13 illustrates the above tree-growing procedure.

The algorithm described above is known as the *Hungarian method*, and

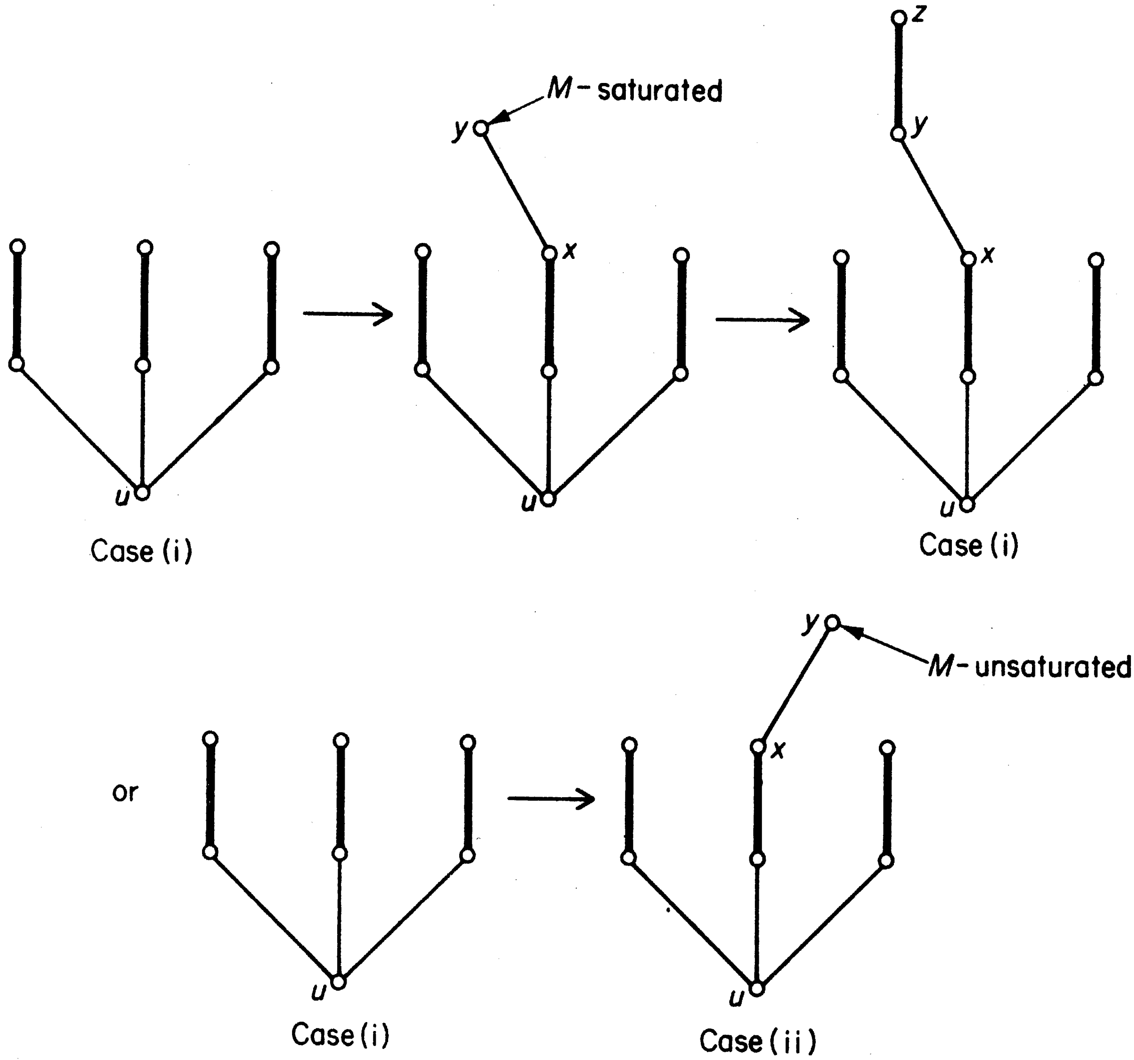


Figure 5.13. The tree-growing procedure

can be summarised as follows:

Start with an arbitrary matching M.
1. If M saturates every vertex in X, stop. Otherwise, let u be an M-unsaturated vertex in X. Set $S=\{u\}$ and $T=\emptyset$.
2. If $N(S)=T$ then $|N(S)|<|S|$, since $|T|=|S|-1$. Stop, since by Hall's theorem there is no matching that saturates every vertex in X. Otherwise, let $y \in N(S)\backslash T$.
3. If y is M-saturated, let $yz \in M$. Replace S by $S \cup \{z\}$ and T by $T \cup \{y\}$ and go to step 2. (Observe that $|T|=|S|-1$ is maintained after this replacement.) Otherwise, let P be an M-augmenting (u, y)-path. Replace M by $\hat{M}=M \,\Delta E(P)$ and go to step 1.

Consider, for example, the graph G in figure 5.14a, with initial matching $M=\{x_2y_2, x_3y_3, x_5y_5\}$. In figure 5.14$b$ an M-alternating tree is grown, starting with x_1, and the M-augmenting path $x_1y_2x_2y_1$ found. This results in a new matching $\hat{M}=\{x_1y_2, x_2y_1, x_3y_3, x_5y_5\}$, and an $\hat{M}$-alternating tree is now grown from x_4 (figures 5.14c and 5.14d) Since there is no $\hat{M}$-augmenting

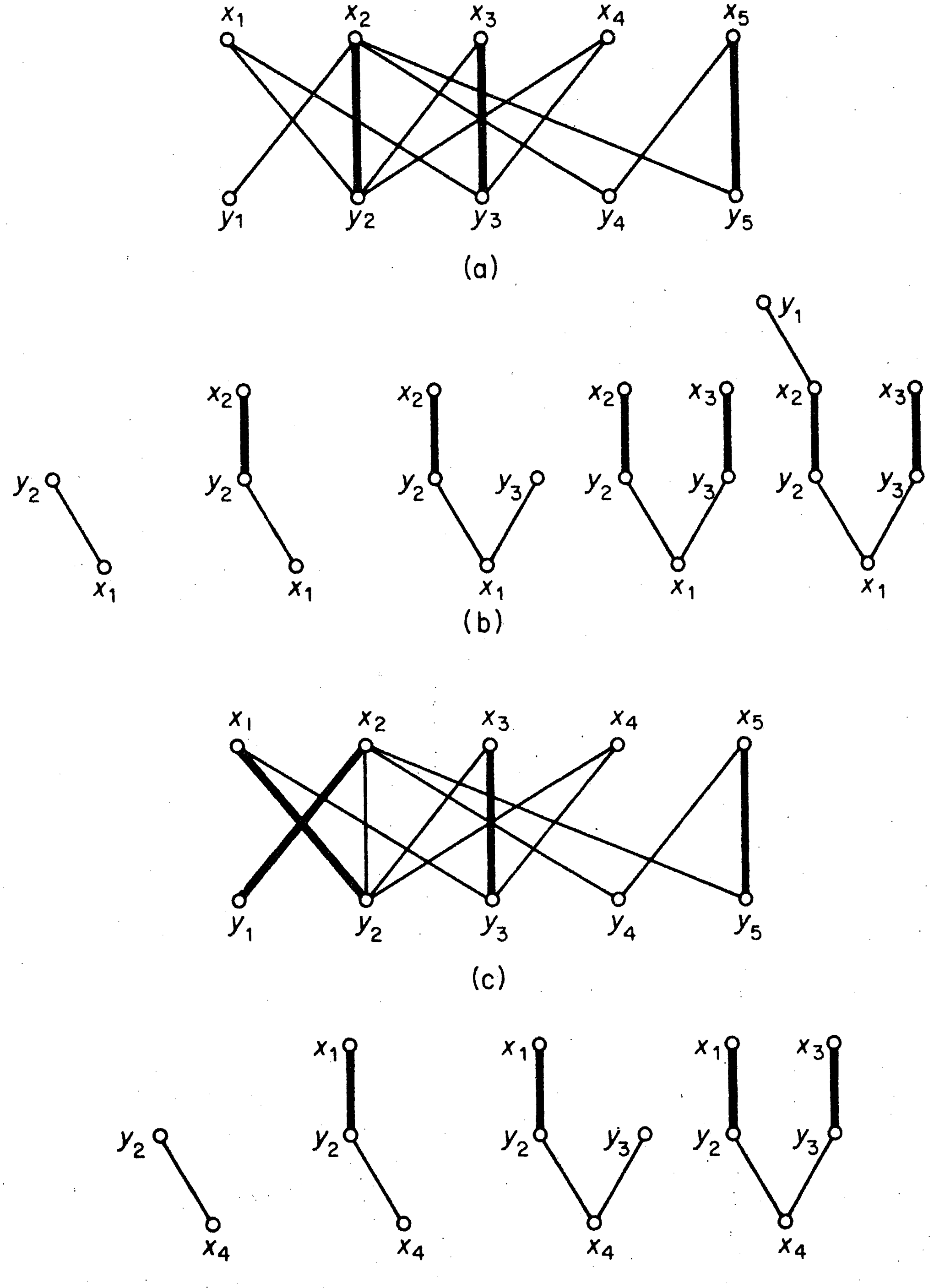


Figure 5.14. (*a*) Matching M; (*b*) an M-alternating tree; (*c*) matching $\hat{M}$; (*d*) an $\hat{M}$-alternating tree

path with origin x_4, the algorithm terminates. The set $S=\{x_1, x_3, x_4\}$, with neighbour set $N(S)=\{y_2, y_3\}$, shows that G has no perfect matching.

A flow diagram of the Hungarian method is given in figure 5.15. Since the algorithm can cycle through the tree-growing procedure, I, at most $|X|$ times before finding either an $S\subseteq X$ such that $|N(S)|<|S|$ or an M-augmenting path, and since the initial matching can be augmented at most $|X|$ times

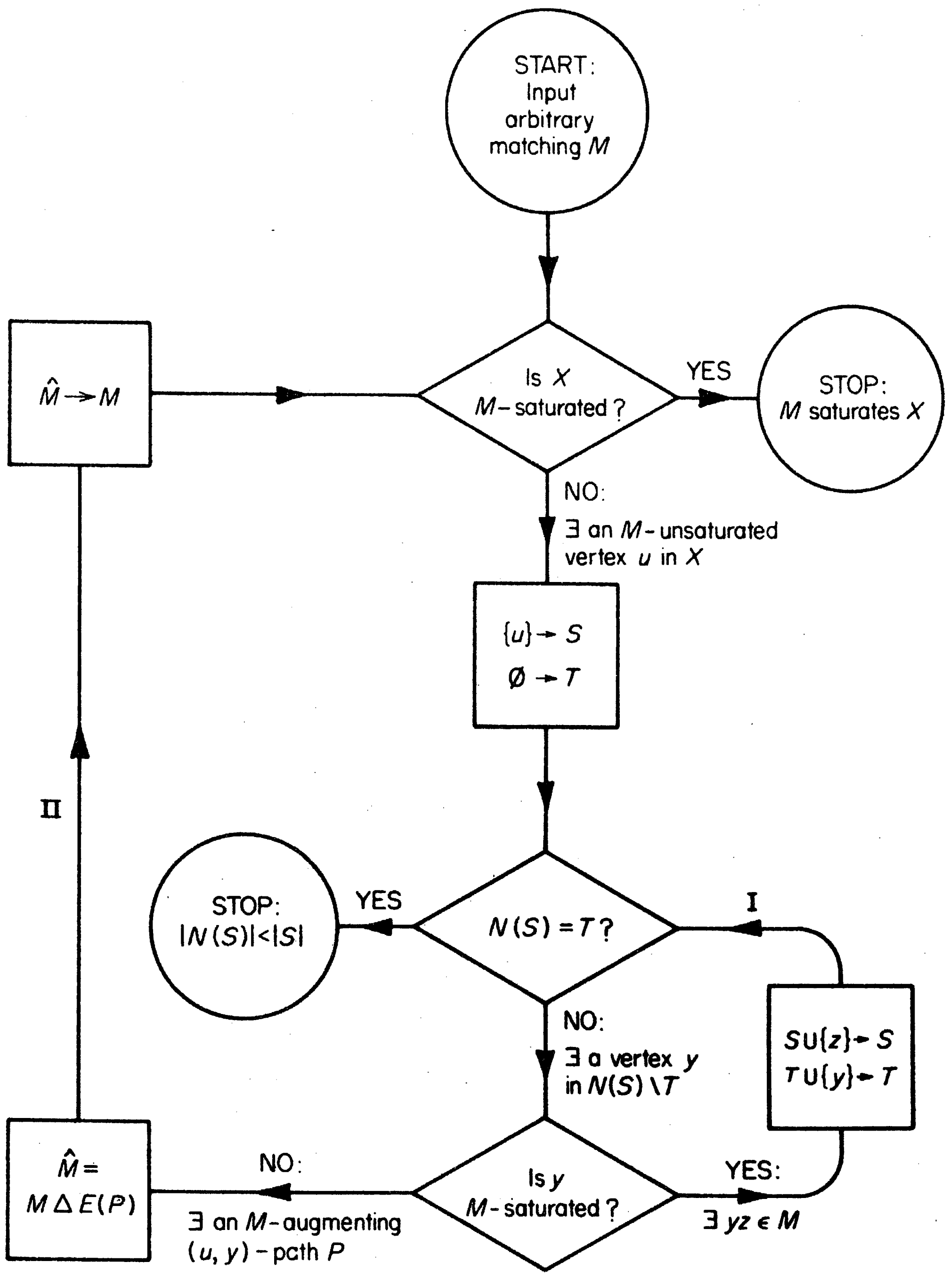


Figure 5.15. The Hungarian method

before a matching of the required type is found, it is clear that the Hungarian method is a good algorithm.

One can find a maximum matching in a bipartite graph by slightly modifying the above procedure (exercise 5.4.1). A good algorithm that determines such a matching in any graph has been given by Edmonds (1965).

Exercise

5.4.1 Describe how the Hungarian method can be used to find a maximum matching in a bipartite graph.

5.5 THE OPTIMAL ASSIGNMENT PROBLEM

The Hungarian method, described in section 5.4, is an efficient way of determining a feasible assignment of workers to jobs, if one exists. However one may, in addition, wish to take into account the effectiveness of the workers in their various jobs (measured, perhaps, by the profit to the company). In this case, one is interested in an assignment that maximises the total effectiveness of the workers. The problem of finding such an assignment is known as the *optimal assignment problem.*

Consider a weighted complete bipartite graph with bipartition (X, Y), where $X=\{x_1, x_2, \ldots, x_n\}$, $Y=\{y_1, y_2, \ldots, y_n\}$ and edge x_iy_j has weight $w_{ij}=w(x_iy_j)$, the effectiveness of worker X_i in job Y_j. The optimal assignment problem is clearly equivalent to that of finding a maximum-weight perfect matching in this weighted graph. We shall refer to such a matching as an *optimal matching.*

To solve the optimal assignment problem it is, of course, possible to enumerate all $n!$ perfect matchings and find an optimal one among them. However, for large n, such a procedure would clearly be most inefficient. In this section we shall present a good algorithm for finding an optimal matching in a weighted complete bipartite graph.

We define a *feasible vertex labelling* as a real-valued function l on the vertex set $X \cup Y$ such that, for all $x \in X$ and $y \in Y$

$$l(x)+l(y) \geq w(xy) \tag{5.11}$$

(The real number $l(v)$ is called the *label* of the vertex v.) A feasible vertex labelling is thus a labelling of the vertices such that the sum of the labels of the two ends of an edge is at least as large as the weight of the edge. No matter what the edge weights are, there always exists a feasible vertex labelling; one such is the function l given by

$$\left.\begin{array}{ll} l(x)=\max\limits_{y \in Y} w(xy) & \text{if } \ x \in X \\ l(y)=0 & \text{if } \ y \in Y \end{array}\right\} \tag{5.12}$$

If l is a feasible vertex labelling, we denote by E_l the set of those edges for which equality holds in (5.11); that is

$$E_l=\{xy \in E \mid l(x)+l(y)=w(xy)\}$$

The spanning subgraph of G with edge set E_l is referred to as the *equality subgraph* corresponding to the feasible vertex labelling l, and is denoted by G_l. The connection between equality subgraphs and optimal matchings is provided by the following theorem.

Theorem 5.5 Let l be a feasible vertex labelling of G. If G_l contains a perfect matching M^*, then M^* is an optimal matching of G.

Proof Suppose that G_l contains a perfect matching M^*. Since G_l is a spanning subgraph of G, M^* is also a perfect matching of G. Now

$$w(M^*)=\sum_{e\in M^*} w(e)=\sum_{v\in V} l(v) \tag{5.13}$$

since each $e\in M^*$ belongs to the equality subgraph and the ends of edges of M^* cover each vertex exactly once. On the other hand, if M is any perfect matching of G, then

$$w(M)=\sum_{e\in M} w(e)\leq \sum_{v\in V} l(v) \tag{5.14}$$

It follows from (5.13) and (5.14) that $w(M^*)\geq w(M)$. Thus M^* is an optimal matching □

The above theorem is the basis of an algorithm, due to Kuhn (1955) and Munkres (1957), for finding an optimal matching in a weighted complete bipartite graph. Our treatment closely follows Edmonds (1967).

Starting with an arbitrary feasible vertex labelling l (for example, the one given in (5.12)), we determine G_l, choose an arbitrary matching M in G_l and apply the Hungarian method. If a perfect matching is found in G_l then, by theorem 5.5, this matching is optimal. Otherwise, the Hungarian method terminates in a matching M' that is not perfect, and an M'-alternating tree H that contains no M'-augmenting path and cannot be grown further (in G_l). We then modify l to a feasible vertex labelling $\hat{l}$ with the property that both M' and H are contained in $G_{\hat{l}}$ and H can be extended in $G_{\hat{l}}$. Such modifications in the feasible vertex labelling are made whenever necessary, until a perfect matching is found in some equality subgraph.

The Kuhn–Munkres Algorithm

Start with an arbitrary feasible vertex labelling l, determine G_l, and choose an arbitrary matching M in G_l.

1. If X is M-saturated, then M is a perfect matching (since $|X|=|Y|$) and hence, by theorem 5.5, an optimal matching; in this case, stop. Otherwise, let u be an M-unsaturated vertex. Set $S=\{u\}$ and $T=\emptyset$.
2. If $N_{G_l}(S)\supset T$, go to step 3. Otherwise, $N_{G_l}(S)=T$. Compute

$$\alpha_l=\min_{\substack{x\in S\\ y\notin T}}\{l(x)+l(y)-w(xy)\}$$

and the feasible vertex labelling $\hat{l}$ given by

$$\hat{l}(v)=\begin{cases} l(v)-\alpha_l & \text{if} \quad v\in S\\ l(v)+\alpha_l & \text{if} \quad v\in T\\ l(v) & \text{otherwise}\end{cases}$$

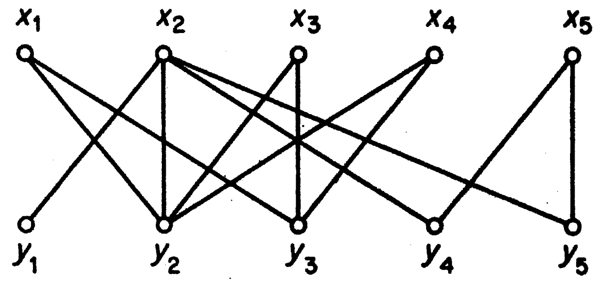
(Note that $\alpha_l>0$ and that $N_{G_{\hat{l}}}(S)\supset T$.) Replace l by $\hat{l}$ and G_l by $G_{\hat{l}}$.

$$\begin{bmatrix} 3 & 5 & 5 & 4 & 1 \\ 2 & 2 & 0 & 2 & 2 \\ 2 & 4 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 & 3 \end{bmatrix}$$

(a)

$$\begin{array}{c} \begin{bmatrix} 3 & \mathbf{5} & \mathbf{5} & 4 & 1 \\ \mathbf{2} & \mathbf{2} & 0 & \mathbf{2} & \mathbf{2} \\ 2 & \mathbf{4} & \mathbf{4} & 1 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 1 & 2 & 1 & \mathbf{3} & \mathbf{3} \end{bmatrix} \begin{matrix} 5 \\ 2 \\ 4 \\ 1 \\ 3 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \end{array}$$

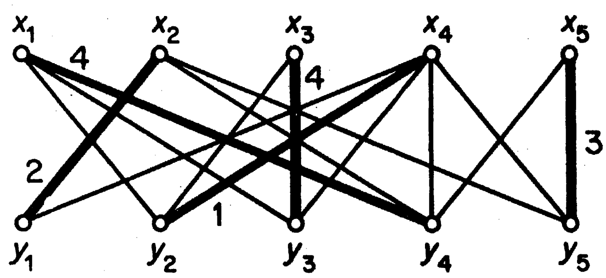
(b)



(c)

$$\begin{array}{c} \begin{bmatrix} 3 & \mathbf{5} & \mathbf{5} & \mathbf{4} & 1 \\ \mathbf{2} & 2 & 0 & \mathbf{2} & \mathbf{2} \\ 2 & \mathbf{4} & \mathbf{4} & 1 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 2 & 1 & \mathbf{3} & \mathbf{3} \end{bmatrix} \begin{matrix} 4 \\ 2 \\ 3 \\ 0 \\ 3 \end{matrix} \\ \begin{matrix} 0 & 1 & 1 & 0 & 0 \end{matrix} \end{array}$$

(d)



(e)

Figure 5.16

3. Choose a vertex y in $N_{G_l}(S)\backslash T$. As in the tree-growing procedure of section 5.4, consider whether or not y is M-saturated. If y is M-saturated, with $yz \in M$, replace S by $S \cup \{z\}$ and T by $T \cup \{y\}$, and go to step 2. Otherwise, let P be an M-augmenting (u, y)-path in G_l, replace M by $\hat{M} = M \,\Delta E(P)$, and go to step 1.

In illustrating the Kuhn–Munkres algorithm, it is convenient to represent a weighted complete bipartite graph G by a matrix $\mathbf{W} = [w_{ij}]$, where w_{ij} is the weight of edge x_iy_j in G. We shall start with the matrix of figure 5.16*a*. In figure 5.16*b*, the feasible vertex labelling (5.12) is shown (by placing the label of x_i to the right of row i of the matrix and the label of y_j below column j) and the entries corresponding to edges of the associated equality subgraph are indicated; the equality subgraph itself is depicted (without weights) in figure 5.16*c*. It was shown in the previous section that this graph has no perfect matching (the set $S = \{x_1, x_3, x_4\}$ has neighbour set $\{y_2, y_3\}$). We therefore modify our initial feasible vertex labelling to the one given in figure 5.16*d*. An application of the Hungarian method now shows that the associated equality subgraph (figure 5.16*e*) has the perfect matching $\{x_1y_4, x_2y_1, x_3y_3, x_4y_2, x_5y_5\}$. This is therefore an optimal matching of G.

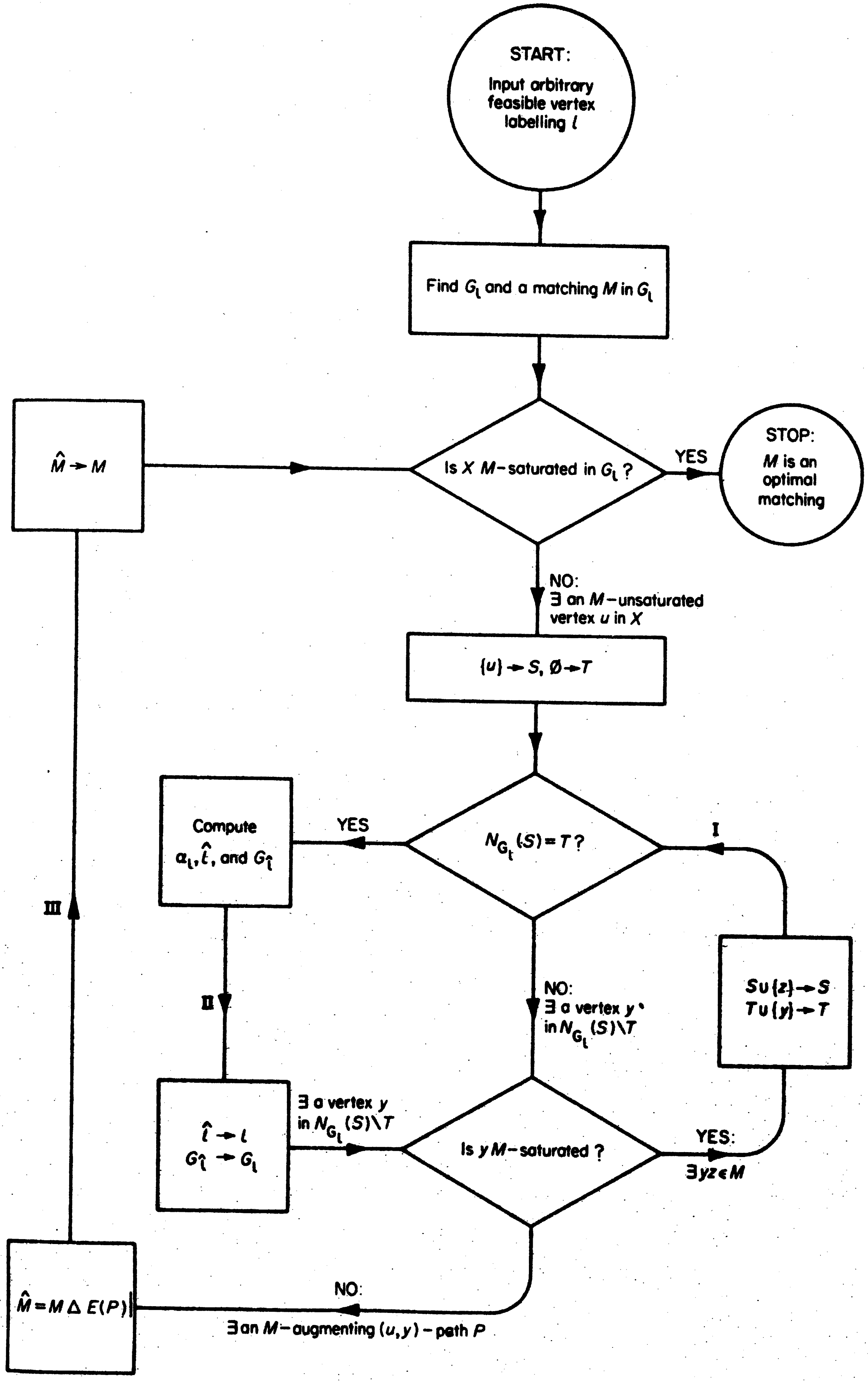


Figure 5.17. The Kuhn–Munkres algorithm

A flow diagram for the Kuhn–Munkres algorithm is given in figure 5.17. In cycle II, the number of computations required to compute $G_{\hat{l}}$ is clearly of order ν^2. Since the algorithm can cycle through I and II at most $|X|$ times before finding an M-augmenting path, and since the initial matching can be augmented at most $|X|$ times before an optimal matching is found, we see that the Kuhn–Munkres algorithm is a good algorithm.

Exercise

5.5.1 A *diagonal* of an $n \times n$ matrix is a set of n entries no two of which belong to the same row or the same column. The weight of a diagonal is the sum of the entries in it. Find a minimum-weight diagonal in the following matrix:

$$\begin{bmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{bmatrix}$$

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6 Edge Colourings

6.1 EDGE CHROMATIC NUMBER

A *k-edge colouring* $\mathscr{C}$ of a loopless graph G is an assignment of k colours, $1, 2, \ldots, k$, to the edges of G. The colouring $\mathscr{C}$ is *proper* if no two adjacent edges have the same colour.

Alternatively, a k-edge colouring can be thought of as a partition $(E_1, E_2, \ldots, E_k)$ of E, where E denotes the (possibly empty) subset of E assigned colour i. A proper k-edge colouring is then a k-edge colouring $(E_1, E_2, \ldots, E_k)$ in which each subset E_i is a matching. The graph of figure 6.1 has the proper 4-edge colouring $(\{a, g\}, \{b, e\}, \{c, f\}, \{d\})$.

G is *k-edge colourable* if G has a proper k-edge-colouring. Trivially, every loopless graph G is ε-edge-colourable; and if G is k-edge-colourable, then G is also l-edge-colourable for every $l > k$. The ***edge chromatic number*** $\chi'(G)$, of a loopless graph G, is the minimum k for which G is k-edge-colourable. G is *k-edge-chromatic* if $\chi'(G) = k$. It can be readily verified that the graph of figure 6.1 has no proper 3-edge colouring. This graph is therefore 4-edge-chromatic.

Clearly, in any proper edge colouring, the edges incident with any one vertex must be assigned different colours. It follows that

$$\chi' \geq \Delta \tag{6.1}$$

Referring to the example of figure 6.1, we see that inequality (6.1) may be strict. However, we shall show that, in the case when G is bipartite, $\chi' = \Delta$. The following simple lemma is basic to our proof. We say that colour i is *represented* at vertex v if some edge incident with v has colour i.

Lemma 6.1.1 Let G be a connected graph that is not an odd cycle. Then

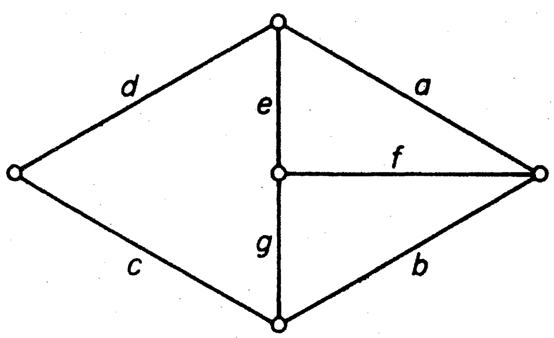


Figure 6.1

G has a 2-edge colouring in which both colours are represented at each vertex of degree at least two.

Proof We may clearly assume that G is nontrivial. Suppose, first, that G is eulerian. If G is an even cycle, the proper 2-edge colouring of G has the required property. Otherwise, G has a vertex v_0 of degree at least four. Let $v_0e_1v_1 \ldots e_\varepsilon v_0$ be an Euler tour of G, and set

$$E_1 = \{e_i \mid i \text{ odd}\} \quad \text{and} \quad E_2 = \{e_i \mid i \text{ even}\} \tag{6.2}$$

Then the 2-edge colouring (E_1, E_2) of G has the required property, since each vertex of G is an internal vertex of $v_0e_1v_1 \ldots e_\varepsilon v_0$.

If G is not eulerian, construct a new graph G^* by adding a new vertex v_0 and joining it to each vertex of odd degree in G. Clearly G^* is eulerian. Let $v_0e_1v_1 \ldots e_{\varepsilon^*} v_0$ be an Euler tour of G^* and define E_1 and E_2 as in (6.2). It is then easily verified that the 2-edge colouring $(E_1 \cap E, E_2 \cap E)$ of G has the required property □

Given a k-edge colouring $\mathscr{C}$ of G we shall denote by $c(v)$ the number of distinct colours represented at v. Clearly, we always have

$$c(v) \leq d(v) \tag{6.3}$$

Moreover, $\mathscr{C}$ is a proper k-edge colouring if and only if equality holds in (6.3) for all vertices v of G. We shall call a k-edge colouring $\mathscr{C}'$ an *improvement* on $\mathscr{C}$ if

$$\sum_{v \in V} c'(v) > \sum_{v \in V} c(v)$$

where $c'(v)$ is the number of distinct colours represented at v in the colouring $\mathscr{C}'$. An *optimal* k-edge colouring is one which cannot be improved.

Lemma 6.1.2 Let $\mathscr{C} = (E_1, E_2, \ldots, E_k)$ be an optimal k-edge colouring of G. If there is a vertex u in G and colours i and j such that i is not represented at u and j is represented at least twice at u, then the component of $G[E_i \cup E_j]$ that contains u is an odd cycle.

Proof Let u be a vertex that satisfies the hypothesis of the lemma, and denote by H the component of $G[E_i \cup E_j]$ containing u. Suppose that H is not an odd cycle. Then, by lemma 6.1.1, H has a 2-edge colouring in which both colours are represented at each vertex of degree at least two in H. When we recolour the edges of H with colours i and j in this way, we obtain a new k-edge colouring $\mathscr{C}' = (E'_1, E'_2, \ldots, E'_k)$ of G. Denoting by $c'(v)$ the number of distinct colours at v in the colouring $\mathscr{C}'$, we have

$$c'(u) = c(u) + 1$$

since, now, both i and j are represented at u, and also

$$c'(v) \geq c(v) \quad \text{for} \quad v \neq u$$

Thus $\sum_{v \in V} c'(v) > \sum_{v \in V} c(v)$, contradicting the choice of $\mathscr{C}$. It follows that H is indeed an odd cycle □

Theorem 6.1 If G is bipartite, then $\chi' = \Delta$.

Proof Let G be a graph with $\chi' > \Delta$, let $\mathscr{C} = (E_1, E_2, \ldots, E_\Delta)$ be an optimal Δ-edge colouring of G, and let u be a vertex such that $c(u) < d(u)$. Clearly, u satisfies the hypothesis of lemma 6.1.2. Therefore G contains an odd cycle and so is not bipartite. It follows from (6.1) that if G is bipartite, then $\chi' = \Delta$ □

An alternative proof of theorem 6.1, using exercise 5.2.3a, is outlined in exercise 6.1.3.

Exercises

6.1.1 Show, by finding an appropriate edge colouring, that $\chi'(K_{m,n}) = \Delta(K_{m,n})$.

6.1.2 Show that the Petersen graph is 4-edge-chromatic.

6.1.3 (*a*) Show that if G is bipartite, then G has a Δ-regular bipartite supergraph.
(*b*) Using (*a*) and exercise 5.2.3a, give an alternative proof of theorem 6.1.

6.1.4 Describe a good algorithm for finding a proper Δ-edge colouring of a bipartite graph G.

6.1.5 Using exercise 1.5.8 and theorem 6.1, show that if G is loopless with $\Delta = 3$, then $\chi' \leq 4$.

6.1.6 Show that if G is bipartite with $\delta > 0$, then G has a δ-edge colouring such that all δ colours are represented at each vertex.
(R. P. Gupta)

6.2 VIZING'S THEOREM

As has already been noted, if G is not bipartite then we cannot necessarily conclude that $\chi' = \Delta$. An important theorem due to Vizing (1964) and, independently, Gupta (1966), asserts that, for any simple graph G, either $\chi' = \Delta$ or $\chi' = \Delta + 1$. The proof given here is by Fournier (1973).

Theorem 6.2 If G is simple, then either $\chi' = \Delta$ or $\chi' = \Delta + 1$.

Proof Let G be a simple graph. By virtue of (6.1) we need only show that $\chi' \leq \Delta + 1$. Suppose, then, that $\chi' > \Delta + 1$. Let $\mathscr{C} = (E_1, E_2, \ldots, E_{\Delta+1})$ be

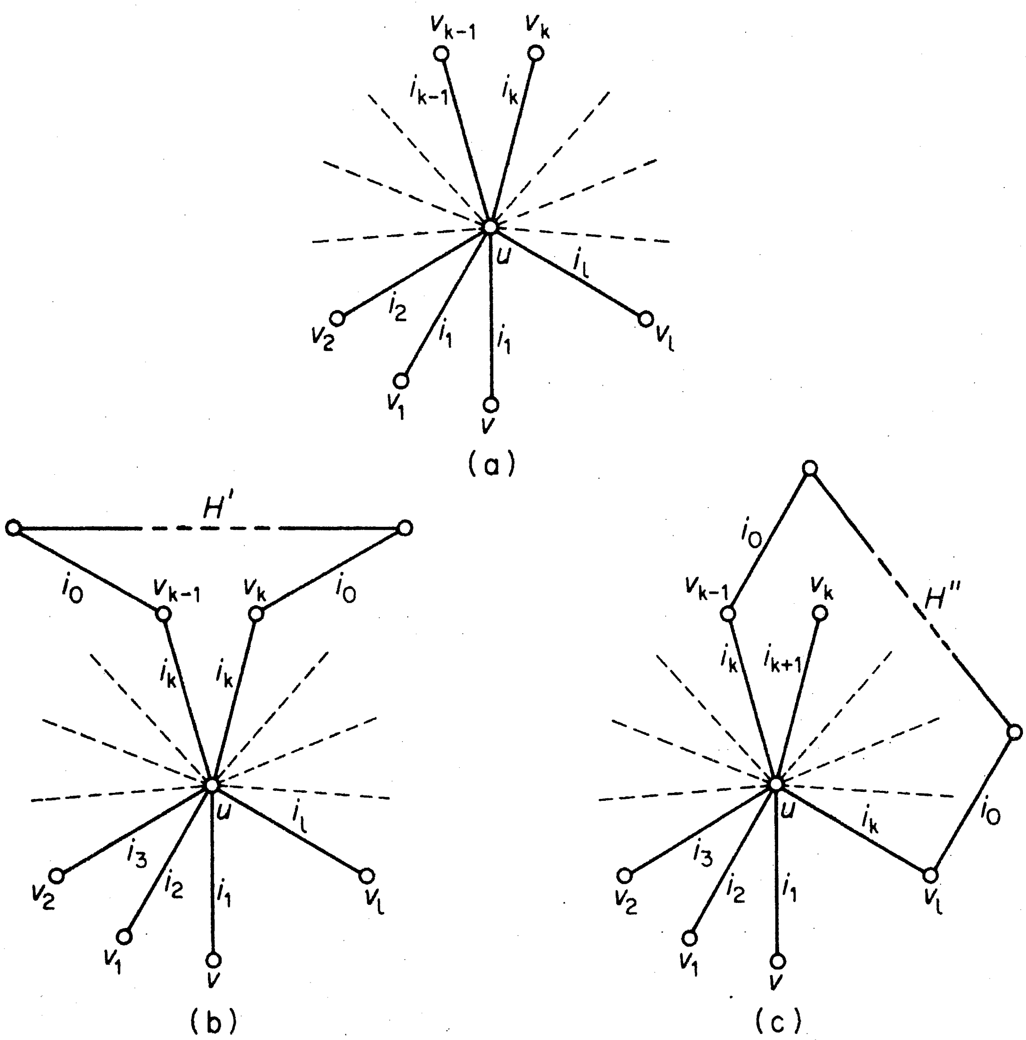


Figure 6.2

an optimal $(\Delta+1)$-edge colouring of G and let u be a vertex such that $c(u) < d(u)$. Then there exist colours i_0 and i_1 such that i_0 is not represented at u, and i_1 is represented at least twice at u. Let uv_1 have colour i_1, as in figure 6.2a.

Since $d(v_1) < \Delta+1$, some colour i_2 is not represented at v_1. Now i_2 must be represented at u since otherwise, by recolouring uv_1 with i_2, we would obtain an improvement on $\mathscr{C}$. Thus some edge uv_2 has colour i_2. Again, since $d(v_2) < \Delta+1$, some colour i_3 is not represented at v_2; and i_3 must be represented at u since otherwise, by recolouring uv_1 with i_2 and uv_2 with i_3, we would obtain an improved $(\Delta+1)$-edge colouring. Thus some edge uv_3 has colour i_3. Continuing this procedure we construct a sequence $v_1, v_2, \ldots$ of vertices and a sequence $i_1, i_2, \ldots$ of colours, such that

(i) uv_j has colour i_j, and
(ii) i_{j+1} is not represented at v_j.

Since the degree of u is finite, there exists a smallest integer l such that, for some $k < l$,

(iii) $i_{l+1} = i_k$.

The situation is depicted in figure 6.2*a*.

We now recolour G as follows. For $1 \le j \le k-1$, recolour uv_j with colour i_{j+1}, yielding a new $(\Delta+1)$-edge colouring $\mathscr{C}' = (E_1', E_2', \ldots, E_{\Delta+1}')$ (figure 6.2*b*). Clearly

$$c'(v) \ge c(v) \quad \text{for all} \quad v \in V$$

and therefore $\mathscr{C}'$ is also an optimal $(\Delta+1)$-edge colouring of G. By lemma 6.1.2, the component H' of $G[E_{i_0}' \cup E_{i_k}']$ that contains u is an odd cycle.

Now, in addition, recolour uv_j with colour i_{j+1}, $k \le j \le l-1$, and uv_l with colour i_k, to obtain a $(\Delta+1)$-edge colouring $\mathscr{C}'' = (E_1'', E_2'', \ldots, E_{\Delta+1}'')$ (figure 6.2*c*). As above

$$c''(v) \ge c(v) \quad \text{for all} \quad v \in V$$

and the component H'' of $G[E_{i_0}'' \cup E_{i_k}'']$ that contains u is an odd cycle. But, since v_k has degree two in H', v_k clearly has degree one in H''. This contradiction establishes the theorem □

Actually, Vizing proved a more general theorem than that given above, one that is valid for all loopless graphs. The maximum number of edges joining two vertices in G is called the *multiplicity* of G, and denoted by $\mu(G)$. We can now state Vizing's theorem in its full generality: if G is loopless, then $\Delta \le \chi' \le \Delta + \mu$.

This theorem is best possible in the sense that, for any μ, there exists a graph G such that $\chi' = \Delta + \mu$. For example, in the graph G of figure 6.3, $\Delta = 2\mu$ and, since any two edges are adjacent, $\chi' = \varepsilon = 3\mu$.

Strong as theorem 6.2 is, it leaves open one interesting question: which simple graphs satisfy $\chi' = \Delta$? The significance of this question will become apparent in chapter 9, when we study edge colourings of planar graphs.

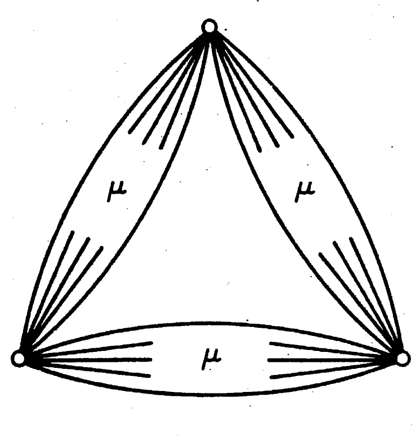


Figure 6.3. A graph G with $\chi' = \Delta + \mu$

Exercises

6.2.1* Show, by finding appropriate edge colourings, that $\chi'(K_{2n-1}) = \chi'(K_{2n}) = 2n-1$.

6.2.2 Show that if G is a nonempty regular simple graph with ν odd, then $\chi' = \Delta + 1$.

6.2.3 (*a*) Let G be a simple graph. Show that if $\nu = 2n+1$ and $\varepsilon > n\Delta$, then $\chi' = \Delta + 1$. (V. G. Vizing)

(*b*) Using (*a*), show that

(i) if G is obtained from a simple regular graph with an even number of vertices by subdividing one edge, then $\chi' = \Delta + 1$;

(ii) if G is obtained from a simple k-regular graph with an odd number of vertices by deleting fewer than $k/2$ edges, then $\chi' = \Delta + 1$. (L. W. Beineke and R. J. Wilson)

6.2.4 (*a*) Show that if G is loopless, then G has a Δ-regular loopless supergraph.

(*b*) Using (*a*) and exercise 5.2.3*b*, show that if G is loopless and Δ is even, then $\chi' \leq 3\Delta/2$.

(Shannon, 1949 has shown that this inequality also holds when Δ is odd.)

6.2.5 G is called *uniquely k-edge-colourable* if any two proper k-edge colourings of G induce the same partition of E. Show that every uniquely 3-edge-colourable 3-regular graph is hamiltonian.

(D. L. Greenwell and H. V. Kronk)

6.2.6 The *product* of simple graphs G and H is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

(*a*) Using Vizing's theorem (6.2), show that $\chi'(G \times K_2) = \Delta(G \times K_2)$.

(*b*) Deduce that if H is nontrivial with $\chi'(H) = \Delta(H)$, then $\chi'(G \times H) = \Delta(G \times H)$.

6.2.7 Describe a good algorithm for finding a proper $(\Delta+1)$-edge colouring of a simple graph G.

6.2.8* Show that if G is simple with $\delta > 1$, then G has a $(\delta - 1)$-edge colouring such that all $\delta - 1$ colours are represented at each vertex.

(R. P. Gupta)

APPLICATIONS

6.3 THE TIMETABLING PROBLEM

In a school, there are m teachers $X_1, X_2, \ldots, X_m$, and n classes $Y_1, Y_2, \ldots, Y_n$. Given that teacher X_i is required to teach class Y_j for p_{ij} periods, schedule a complete timetable in the minimum possible number of periods.

The above problem is known as the *timetabling problem*, and can be solved completely using the theory of edge colourings developed in this chapter. We represent the teaching requirements by a bipartite graph G with bipartition (X, Y), where $X=\{x_1, x_2, \ldots, x_m\}$, $Y=\{y_1, y_2, \ldots, y_n\}$ and vertices x_i and y_j are joined by p_{ij} edges. Now, in any one period, each teacher can teach at most one class, and each class can be taught by at most one teacher—this, at least, is our assumption. Thus a teaching schedule for one period corresponds to a matching in the graph and, conversely, each matching corresponds to a possible assignment of teachers to classes for one period. Our problem, therefore, is to partition the edges of G into as few matchings as possible or, equivalently, to properly colour the edges of G with as few colours as possible. Since G is bipartite, we know, by theorem 6.1, that $\chi'=\Delta$. Hence, if no teacher teaches for more than p periods, and if no class is taught for more than p periods, the teaching requirements can be scheduled in a p-period timetable. Furthermore, there is a good algorithm for constructing such a timetable, as is indicated in exercise 6.1.4. We thus have a complete solution to the timetabling problem.

However, the situation might not be so straightforward. Let us assume that only a limited number of classrooms are available. With this additional constraint, how many periods are now needed to schedule a complete timetable?

Suppose that altogether there are l lessons to be given, and that they have been scheduled in a p-period timetable. Since this timetable requires an average of l/p lessons to be given per period, it is clear that at least $\{l/p\}$ rooms will be needed in some one period. It turns out that one can always arrange l lessons in a p-period timetable so that at most $\{l/p\}$ rooms are occupied in any one period. This follows from theorem 6.3 below. We first have a lemma.

Lemma 6.3 Let M and N be disjoint matchings of G with $|M|>|N|$. Then there are disjoint matchings M' and N' of G such that $|M'|=|M|-1$, $|N'|=|N|+1$ and $M'\cup N'=M\cup N$.

Proof Consider the graph $H=G[M\cup N]$. As in the proof of theorem 5.1, each component of H is either an even cycle, with edges alternately in M and N, or else a path with edges alternately in M and N. Since $|M|>|N|$, some path component P of H must start and end with edges of M. Let $P=v_0e_1v_1\ldots e_{2n+1}v_{2n+1}$, and set

$$M'=(M\backslash\{e_1, e_3, \ldots, e_{2n+1}\})\cup\{e_2, e_4, \ldots, e_{2n}\}$$
$$N'=(N\backslash\{e_2, e_4, \ldots, e_{2n}\})\cup\{e_1, e_3, \ldots, e_{2n+1}\}$$

Then M' and N' are matchings of G that satisfy the conditions of the lemma $\square$

$$\mathbf{P} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{array} \begin{array}{c} \begin{array}{ccccc} Y_1 & Y_2 & Y_3 & Y_4 & Y_5 \end{array} \\ \left[\begin{array}{ccccc} 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

(a)

	Period 1	2	3	4
X_1	Y_1	Y_1	Y_3	Y_4
X_2	Y_2	–	Y_4	–
X_3	Y_3	Y_4	–	Y_2
X_4	Y_4	Y_5	–	–

(b)

Figure 6.4

Theorem 6.3 **If G is bipartite, and if $p \geq \Delta$, then there exist p disjoint matchings $M_1, M_2, \ldots, M_p$ of G such that**

$$E = M_1 \cup M_2 \cup \ldots \cup M_p \tag{6.4}$$

and, for $1 \leq i \leq p$

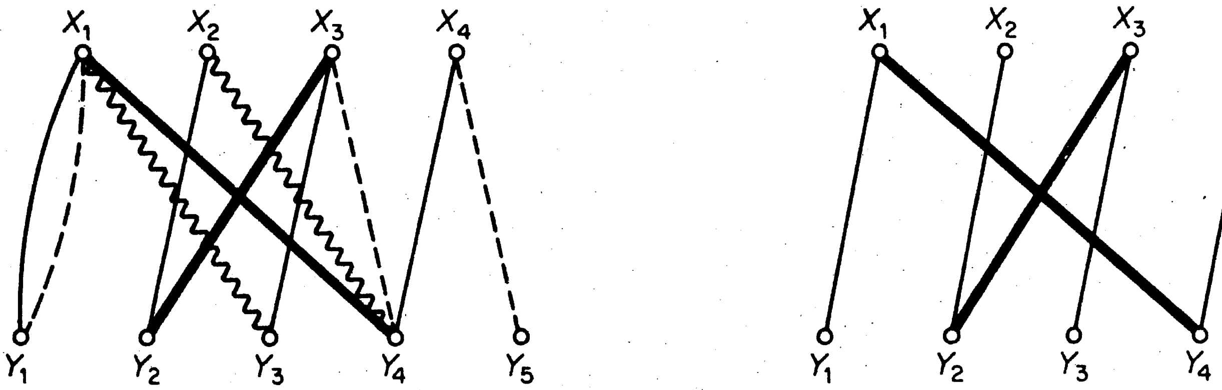
$$[\varepsilon/p] \leq |M_i| \leq \{\varepsilon/p\} \tag{6.5}$$

(Note: condition (6.5) says that any two matchings M_i and M_j differ in size by at most one.)

Proof Let G be a bipartite graph. By theorem 6.1, the edges of G can be partitioned into Δ matchings $M'_1, M'_2, \ldots, M'_\Delta$. Therefore, for any $p \geq \Delta$, there exist p disjoint matchings $M'_1, M'_2, \ldots, M'_p$ (with $M'_i = \emptyset$ for $i > \Delta$) such that

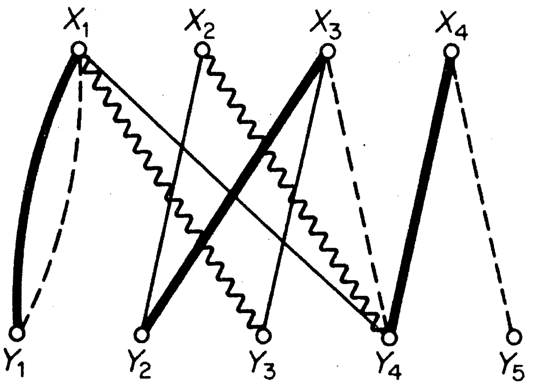
$$E = M'_1 \cup M'_2 \cup \ldots \cup M'_p$$

By repeatedly applying lemma 6.3 to pairs of these matchings that differ in size by more than one, we eventually obtain p disjoint matchings $M_1, M_2, \ldots, M_p$ of G satisfying (6.4) and (6.5), as required □



(a) (b)

Figure 6.5



(a)

	Period 1	2	3	4
x_1	y_4	y_1	y_3	y_1
x_2	y_2	–	y_4	–
x_3	y_3	y_4	–	y_2
x_4	–	y_5	–	y_4

(b)

Figure 6.6

As an example, suppose that there are four teachers and five classes, and that the teaching requirement matrix $\mathbf{P}=[p_{ij}]$ is as given in figure 6.4*a*. One possible 4-period timetable is shown in figure 6.4*b*.

We can represent the above timetable by a decomposition into matchings of the edge set of the bipartite graph G corresponding to $\mathbf{P}$, as shown in figure 6.5*a*. (Normal edges correspond to period 1, broken edges to period 2, wavy edges to period 3, and heavy edges to period 4.)

From the timetable we see that four classes are taught in period 1, and so four rooms are needed. However $\varepsilon = 11$ and so, by theorem 6.4, a 4-period timetable can be arranged so that in each period either $2(=[11/4])$ or $3(=\{11/4\})$ classes are taught. Let M_1 denote the normal matching and M_4 the heavy matching; notice that $|M_1|=4$ and $|M_4|=2$. We can now find a 4-period 3-room timetable by considering $G[M_1 \cup M_4]$ (figure 6.5*b*). $G[M_1 \cup M_4]$ has two components, each consisting of a path of length three. Both paths start and end with normal edges and so, by interchanging the matchings on one of the two paths, we shall reduce the normal matching to one of three edges, and at the same time increase the heavy matching to one of three edges. If we choose the path $y_1x_1y_4x_4$, making the edges y_1x_1 and y_4x_4 heavy and the edge x_1y_4 normal, we obtain the decomposition of E shown in figure 6.6*a*. This then gives the revised timetable shown in figure 6.6*b*; here, only three rooms are needed at any one time.

	Period 1	2	3	4	5	6
x_1	y_4	y_3	y_1	–	y_1	–
x_2	y_2	y_4	–	–	–	–
x_3	–	–	y_4	y_3	y_2	–
x_4	–	–	–	y_4	–	y_5

Figure 6.7

However, suppose that there are just two rooms available. Theorem 6.4 tells us that there must be a 6-period timetable that satisfies our requirements (since $\{11/6\}=2$). Such a timetable is given in figure 6.7.

In practice, most problems on timetabling are complicated by preassignments (that is, conditions specifying the periods during which certain teachers and classes must meet). This generalisation of the timetabling problem has been studied by Dempster (1971) and de Werra (1970).

Exercise

6.3.1 In a school there are seven teachers and twelve classes. The teaching requirements for a five-day week are given by the matrix

$\mathbf{P} =$

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}
X_1	3	2	3	3	3	3	3	3	3	3	3	3
X_2	1	3	6	0	4	2	5	1	3	3	0	4
X_3	5	0	5	5	0	0	5	0	5	0	5	5
X_4	2	4	2	4	2	4	2	4	2	4	2	3
X_5	3	5	2	2	0	3	1	4	4	3	2	5
X_6	5	5	0	0	5	5	0	5	0	5	5	0
X_7	0	3	4	3	4	3	4	3	4	3	3	0

where p_{ij} is the number of periods that teacher X_i must teach class Y_j..

(*a*) Into how many periods must a day be divided so that the requirements can be satisfied?

(*b*) If an eight-period/day timetable is drawn up, how many classrooms will be needed?

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7 Independent Sets and Cliques

7.1 INDEPENDENT SETS

A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. An independent set is *maximum* if G has no independent set S' with $|S'|>|S|$. Examples of independent sets are shown in figure 7.1.

Recall that a subset K of V such that every edge of G has at least one end in K is called a covering of G. The two examples of independent sets given in figure 7.1 are both complements of coverings. It is not difficult to see that this is always the case.

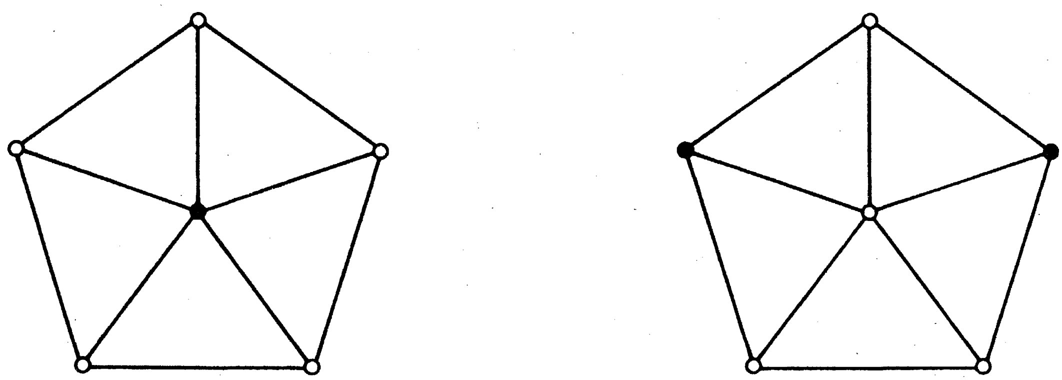
Theorem 7.1 A set $S \subseteq V$ is an independent set of G if and only if $V \backslash S$ is a covering of G.

Proof By definition, S is an independent set of G if and only if no edge of G has both ends in S or, equivalently, if and only if each edge has at least one end in $V \backslash S$. But this is so if and only if $V \backslash S$ is a covering of G □

The number of vertices in a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$; similarly, the number of vertices in a minimum covering of G is the *covering number* of G and is denoted by $\beta(G)$.

Corollary 7.1 $\alpha+\beta=\nu$.

Proof Let S be a maximum independent set of G, and let K be a minimum covering of G. Then, by theorem 7.1, $V \backslash K$ is an independent set



(a) (b)

Figure 7.1. (*a*) An independent set; (*b*) a maximum independent set

and $V\backslash S$ is a covering. Therefore

$$\nu - \beta = |V\backslash K| \leq \alpha \tag{7.1}$$

and

$$\nu - \alpha = |V\backslash S| \geq \beta \tag{7.2}$$

Combining (7.1) and (7.2) we have $\alpha + \beta = \nu$ □

The edge analogue of an independent set is a set of links no two of which are adjacent, that is, a matching. The edge analogue of a covering is called an edge covering. An *edge covering* of G is a subset L of E such that each vertex of G is an end of some edge in L. Note that edge coverings do not always exist; a graph G has an edge covering if and only if $\delta > 0$. We denote the number of edges in a maximum matching of G by $\alpha'(G)$, and the number of edges in a minimum edge covering of G by $\beta'(G)$; the numbers $\alpha'(G)$ and $\beta'(G)$ are the *edge independence number* and *edge covering number* of G, respectively.

Matchings and edge coverings are not related to one another as simply as are independent sets and coverings; the complement of a matching need not be an edge covering, nor is the complement of an edge covering necessarily a matching. However, it so happens that the parameters α' and β' are related in precisely the same manner as are α and β.

Theorem 7.2 (Gallai, 1959) If $\delta > 0$, then $\alpha' + \beta' = \nu$.

Proof Let M be a maximum matching in G and let U be the set of M-unsaturated vertices. Since $\delta > 0$ and M is maximum, there exists a set E' of $|U|$ edges, one incident with each vertex in U. Clearly, $M \cup E'$ is an edge covering of G, and so

$$\beta' \leq |M \cup E'| = \alpha' + (\nu - 2\alpha') = \nu - \alpha'$$

or

$$\alpha' + \beta' \leq \nu \tag{7.3}$$

Now let L be a minimum edge covering of G, set $H = G[L]$ and let M be a maximum matching in H. Denote the set of M-unsaturated vertices in H by U. Since M is maximum, $H[U]$ has no links and therefore

$$|L| - |M| = |L\backslash M| \geq |U| = \nu - 2\,|M|$$

Because H is a subgraph of G, M is a matching in G and so

$$\alpha' + \beta' \geq |M| + |L| \geq \nu \tag{7.4}$$

Combining (7.3) and (7.4), we have $\alpha' + \beta' = \nu$ □

We can now prove a theorem that bears a striking formal resemblance to König's theorem (5.3).

Theorem 7.3 In a bipartite graph G with $\delta > 0$, the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering.

Proof Let G be a bipartite graph with $\delta > 0$. By corollary 7.1 and theorem 7.2, we have

$$\alpha + \beta = \alpha' + \beta'$$

and, since G is bipartite, it follows from theorem 5.3 that $\alpha' = \beta$. Thus $\alpha = \beta'$ □

Even though the concept of an independent set is analogous to that of a matching, there exists no theory of independent sets comparable to the theory of matchings presented in chapter 5; for example, no good algorithm for finding a maximum independent set in a graph is known. However, there are two interesting theorems that relate the number of vertices in a maximum independent set of a graph to various other parameters of the graph. These theorems will be discussed in sections 7.2 and 7.3.

Exercises

7.1.1 (*a*) Show that G is bipartite if and only if $\alpha(H) \geq \frac{1}{2}\nu(H)$ for every subgraph H of G.
(*b*) Show that G is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph H of G such that $\delta(H) > 0$.

7.1.2 A graph is *α-critical* if $\alpha(G-e) > \alpha(G)$ for all $e \in E$. Show that a connected α-critical graph has no cut vertices.

7.1.3 A graph G is *β-critical* if $\beta(G-e) < \beta(G)$ for all $e \in E$. Show that
(*a*) a connected β-critical graph has no cut vertices;
(*b*)* if G is connected, then $\beta \leq \frac{1}{2}(\varepsilon + 1)$.

7.2 RAMSEY'S THEOREM

In this section we deal only with simple graphs. A *clique* of a simple graph G is a subset S of V such that $G[S]$ is complete. Clearly, S is a clique of G if and only if S is an independent set of G^c, and so the two concepts are complementary.

If G has no large cliques, then one might expect G to have a large independent set. That this is indeed the case was first proved by Ramsey (1930). He showed that, given any positive integers k and l, there exists a smallest integer $r(k, l)$ such that every graph on $r(k, l)$ vertices contains either a clique of k vertices or an independent set of l vertices. For example, it is easy to see that

$$r(1, l) = r(k, 1) = 1 \qquad (7.5)$$

and

$$r(2, l) = l, \qquad r(k, 2) = k \tag{7.6}$$

The numbers $r(k, l)$ are known as the *Ramsey numbers*. The following theorem on Ramsey numbers is due to Erdös and Szekeres (1935) and Greenwood and Gleason (1955).

Theorem 7.4 For any two integers $k \geq 2$ and $l \geq 2$

$$r(k, l) \leq r(k, l-1) + r(k-1, l) \tag{7.7}$$

Furthermore, if $r(k, l-1)$ and $r(k-1, l)$ are both even, then strict inequality holds in (7.7).

Proof Let G be a graph on $r(k, l-1) + r(k-1, l)$ vertices, and let $v \in V$. We distinguish two cases:

(i) v is nonadjacent to a set S of at least $r(k, l-1)$ vertices, or
(ii) v is adjacent to a set T of at least $r(k-1, l)$ vertices.

Note that either case (i) or case (ii) must hold because the number of vertices to which v is nonadjacent plus the number of vertices to which v is adjacent is equal to $r(k, l-1) + r(k-1, l) - 1$.

In case (i), $G[S]$ contains either a clique of k vertices or an independent set of $l-1$ vertices, and therefore $G[S \cup \{v\}]$ contains either a clique of k vertices or an independent set of l vertices. Similarly, in case (ii), $G[T \cup \{v\}]$ contains either a clique of k vertices or an independent set of l vertices. Since one of case (i) and case (ii) must hold, it follows that G contains either a clique of k vertices or an independent set of l vertices. This proves (7.7).

Now suppose that $r(k, l-1)$ and $r(k-1, l)$ are both even, and let G be a graph on $r(k, l-1) + r(k-1, l) - 1$ vertices. Since G has an odd number of vertices, it follows from corollary 1.1 that some vertex v is of even degree; in particular, v cannot be adjacent to precisely $r(k-1, l) - 1$ vertices. Consequently, either case (i) or case (ii) above holds, and therefore G contains either a clique of k vertices or an independent set of l vertices. Thus

$$r(k, l) \leq r(k, l-1) + r(k-1, l) - 1$$

as stated □

The determination of the Ramsey numbers in general is a very difficult unsolved problem. Lower bounds can be obtained by the construction of suitable graphs. Consider, for example, the four graphs in figure 7.2.

The 5-cycle (figure 7.2*a*) contains no clique of three vertices and no independent set of three vertices. It shows, therefore, that

$$r(3, 3) \geq 6 \tag{7.8}$$

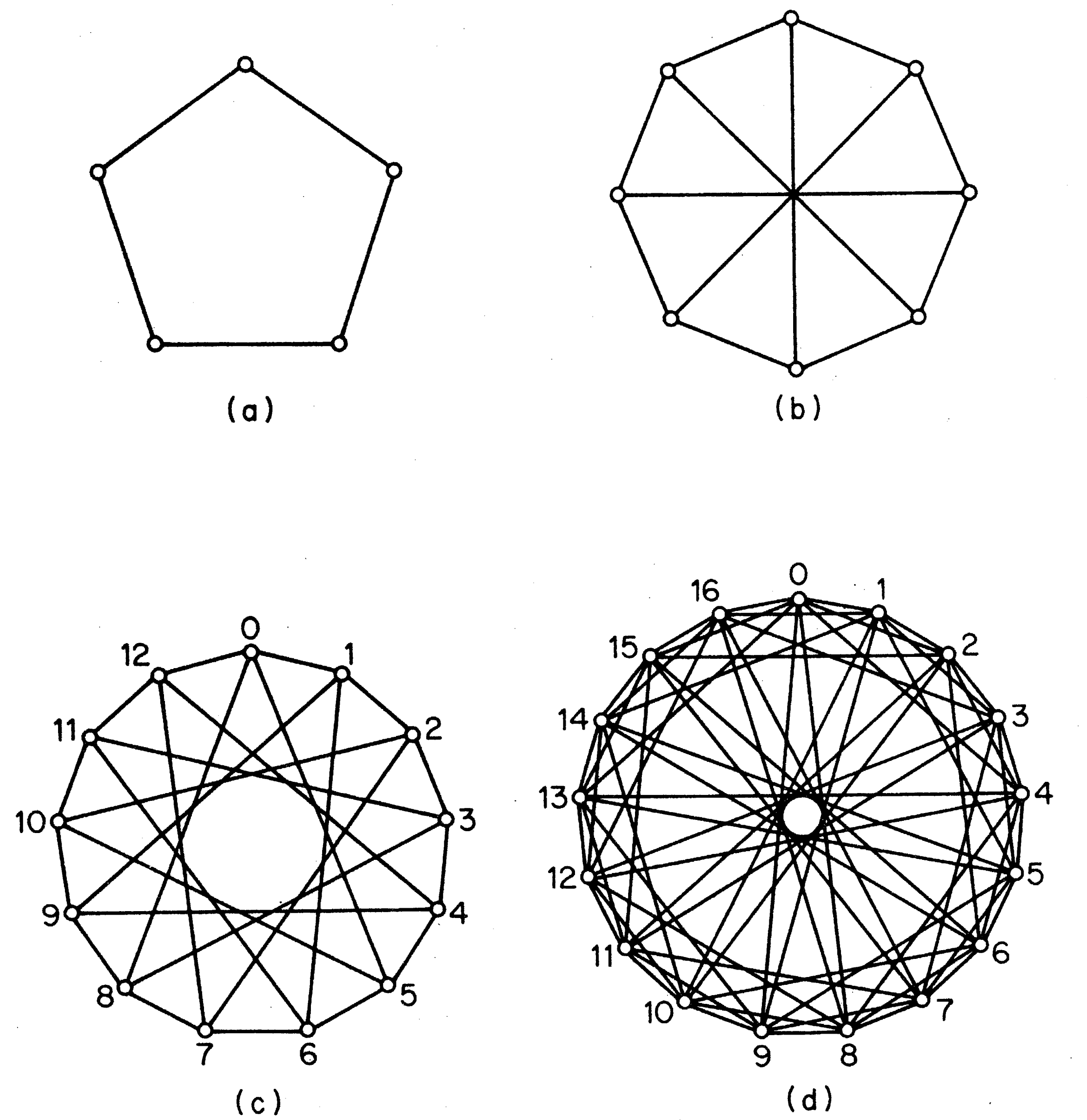


Figure 7.2. (*a*) A (3,3)-Ramsey graph; (*b*) a (3,4)-Ramsey graph; (*c*) a (3,5)-Ramsey graph; (*d*) a (4,4)-Ramsey graph

The graph of figure 7.2*b* contains no clique of three vertices and no independent set of four vertices. Hence

$$r(3, 4) \geq 9 \tag{7.9}$$

Similarly, the graph of figure 7.2*c* shows that

$$r(3, 5) \geq 14 \tag{7.10}$$

and the graph of figure 7.2*d* yields

$$r(4, 4) \geq 18 \tag{7.11}$$

With the aid of theorem 7.4 and equations (7.6) we can now show that equality in fact holds in (7.8), (7.9), (7.10) and (7.11). Firstly, by (7.7) and (7.6)

$$r(3, 3) \leq r(3, 2) + r(2, 3) = 6$$

and therefore, using (7.8), we have $r(3,3)=6$. Noting that $r(3,3)$ and $r(2,4)$ are both even, we apply theorem 7.4 and (7.6) to obtain

$$r(3,4)\le r(3,3)+r(2,4)-1=9$$

With (7.9) this gives $r(3,4)=9$. Now we again apply (7.7) and (7.6) to obtain

$$r(3,5)\le r(3,4)+r(2,5)=14$$

and

$$r(4,4)\le r(4,3)+r(3,4)=18$$

which, together with (7.10) and (7.11), respectively, yield $r(3,5)=14$ and $r(4,4)=18$.

The following table shows all Ramsey numbers $r(k, l)$ known to date.

$k \backslash l$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	9	14	18	23
4	1	4	9	18			

A (k, l)-*Ramsey graph* is a graph on $r(k,l)-1$ vertices that contains neither a clique of k vertices nor an independent set of l vertices. By definition of $r(k, l)$ such graphs exist for all $k\ge 2$ and $l\ge 2$. Ramsey graphs often seem to possess interesting structures. All of the graphs in figure 7.2 are Ramsey graphs; the last two can be obtained from finite fields in the following way. We get the (3, 5)-Ramsey graph by regarding the thirteen vertices as elements of the field of integers modulo 13, and joining two vertices by an edge if their difference is a cubic residue of 13 (either 1, 5, 8 or 12); the (4, 4)-Ramsey graph is obtained by regarding the vertices as elements of the field of integers modulo 17, and joining two vertices if their difference is a quadratic residue of 17 (either 1, 2, 4, 8, 9, 13, 15 or 16). It has been conjectured that the (k, k)-Ramsey graphs are always self-complementary (that is, isomorphic to their complements); this is true for $k=2$, 3 and 4.

In general, theorem 7.4 yields the following upper bound for $r(k, l)$.

Theorem 7.5 $r(k,l)\le\binom{k+l-2}{k-1}$

Proof By induction on $k+l$. Using (7.5) and (7.6) we see that the theorem holds when $k+l\le 5$. Let m and n be positive integers, and assume that the theorem is valid for all positive integers k and l such that

$5 \le k+l < m+n$. Then, by theorem 7.4 and the induction hypothesis

$$r(m, n) \le r(m, n-1) + r(m-1, n) \le \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2} = \binom{m+n-2}{m-1}$$

Thus the theorem holds for all values of k and l □

A lower bound for $r(k, k)$ is given in the next theorem. It is obtained by means of a powerful technique known as the *probabilistic method* (see Erdös and Spencer, 1974). The probabilistic method is essentially a crude counting argument. Although nonconstructive, it can often be applied to assert the existence of a graph with certain specified properties.

Theorem 7.6 (Erdös, 1947) $r(k, k) \ge 2^{k/2}$

Proof. Since $r(1, 1) = 1$ and $r(2, 2) = 2$, we may assume that $k \ge 3$. Denote by $\mathcal{G}_n$ the set of simple graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$, and by $\mathcal{G}_n^k$ the set of those graphs in $\mathcal{G}_n$ that have a clique of k vertices. Clearly

$$|\mathcal{G}_n| = 2^{\binom{n}{2}} \tag{7.12}$$

since each subset of the $\binom{n}{2}$ possible edges v_iv_j determines a graph in $\mathcal{G}_n$. Similarly, the number of graphs in $\mathcal{G}_n$ having a particular set of k vertices as a clique is $2^{\binom{n}{2}-\binom{k}{2}}$. Since there are $\binom{n}{k}$ distinct k-element subsets of $\{v_1, v_2, \ldots, v_n\}$, we have

$$|\mathcal{G}_n^k| \le \binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}} \tag{7.13}$$

By (7.12) and (7.13)

$$\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} \le \binom{n}{k} 2^{-\binom{k}{2}} < \frac{n^k 2^{-\binom{k}{2}}}{k!} \tag{7.14}$$

Suppose, now, that $n < 2^{k/2}$. From (7.14) it follows that

$$\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} < \frac{2^{k^2/2} 2^{-\binom{k}{2}}}{k!} = \frac{2^{k/2}}{k!} < \tfrac{1}{2}$$

Therefore, fewer than half of the graphs in $\mathcal{G}_n$ contain a clique of k vertices. Also, because $\mathcal{G}_n = \{G \mid G^c \in \mathcal{G}_n\}$, fewer than half of the graphs in $\mathcal{G}_n$ contain an independent set of k vertices. Hence some graph in $\mathcal{G}_n$ contains neither a clique of k vertices nor an independent set of k vertices. Because this holds for any $n < 2^{k/2}$, we have $r(k, k) \ge 2^{k/2}$ □

From theorem 7.6 we can immediately deduce a lower bound for $r(k, l)$.

Corollary 7.6 If $m = \min\{k, l\}$, then $r(k, l) \geq 2^{m/2}$

All known lower bounds for $r(k, l)$ obtained by constructive arguments are much weaker than that given in corollary 7.6; the best is due to Abbott (1972), who shows that $r(2^n+1, 2^n+1) \geq 5^n+1$ (exercise 7.2.4).

The Ramsey numbers $r(k, l)$ are sometimes defined in a slightly different way from that given at the beginning of this section. One easily sees that $r(k, l)$ can be thought of as the smallest integer n such that every 2-edge colouring (E_1, E_2) of K_n contains either a complete subgraph on k vertices, all of whose edges are in colour 1, or a complete subgraph on l vertices, all of whose edges are in colour 2. Expressed in this form, the Ramsey numbers have a natural generalisation. We define $r(k_1, k_2, \ldots, k_m)$ to be the smallest integer n such that every m-edge colouring $(E_1, E_2, \ldots, E_m)$ of K_n contains, for some i, a complete subgraph on k_i vertices, all of whose edges are in colour i.

The following theorem and corollary generalise (7.7) and theorem 7.5, and can be proved in a similar manner. They are left as an exercise (7.2.2).

Theorem 7.7 $r(k_1, k_2, \ldots, k_m) \leq r(k_1-1, k_2, \ldots, k_m) + r(k_1, k_2-1, \ldots, k_m) + \ldots + r(k_1, k_2, \ldots, k_m-1) - m + 2$

Corollary 7.7 $r(k_1+1, k_2+1, \ldots, k_m+1) \leq \dfrac{(k_1+k_2+\ldots+k_m)!}{k_1!\,k_2!\ldots k_m!}$

Exercises

7.2.1 Show that, for all k and l, $r(k, l) = r(l, k)$.

7.2.2 Prove theorem 7.7 and corollary 7.7.

7.2.3 Let r_n denote the Ramsey number $r(k_1, k_2, \ldots, k_n)$ with $k_i = 3$ for all i.

(*a*) Show that $r_n \leq n(r_{n-1}-1)+2$.
(*b*) Noting that $r_2 = 6$, use (*a*) to show that $r_n \leq [n!\,e]+1$.
(*c*) Deduce that $r_3 \leq 17$.
(Greenwood and Gleason, 1955 have shown that $r_3 = 17$.)

7.2.4 The *composition* of simple graphs G and H is the simple graph $G[H]$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

(*a*) Show that $\alpha(G[H]) \leq \alpha(G)\alpha(H)$.
(*b*) Using (*a*), show that

$$r(kl+1, kl+1) - 1 \geq (r(k+1, k+1)-1) \times (r(l+1, l+1)-1)$$

(*c*) Deduce that $r(2^n+1, 2^n+1) \geq 5^n+1$ for all $n \geq 0$.

(H. L. Abbott)

7.2.5 Show that the join of a 3-cycle and a 5-cycle contains no K_6, but that every 2-edge colouring yields a monochromatic triangle.

(R. L. Graham)

(Folkman, 1970 has constructed a graph containing no K_4 in which every 2-edge colouring yields a monochromatic triangle—this graph has a very large number of vertices.)

7.2.6 Let $G_1, G_2, \ldots, G_m$ be simple graphs. The *generalised Ramsey number* $r(G_1, G_2, \ldots, G_m)$ is the smallest integer n such that every m-edge colouring $(E_1, E_2, \ldots, E_m)$ of K_n contains, for some i, a subgraph isomorphic to G_i in colour i. Show that

(*a*) if G is a path of length three and H is a 4-cycle, then $r(G, G) = 5$, $r(G, H) = 5$ and $r(H, H) = 6$;

(*b*)* if T is any tree on m vertices and if $m-1$ divides $n-1$, then $r(T, K_{1,n}) = m+n-1$;

(*c*)* if T is any tree on m vertices, then $r(T, K_n) = (m-1)(n-1)+1$.

(V. Chvátal)

7.3 TURÁN'S THEOREM

In this section, we shall prove a well-known theorem due to Turán (1941). It determines the maximum number of edges that a simple graph on ν vertices can have without containing a clique of size $m+1$. Turán's theorem has become the basis of a significant branch of graph theory known as *extremal graph theory* (see Erdös, 1967). We shall derive it from the following result of Erdös (1970).

Theorem 7.8 If a simple graph G contains no K_{m+1}, then G is degree-majorised by some complete m-partite graph H. Moreover, if G has the same degree sequence as H, then $G \cong H$.

Proof By induction on m. The theorem is trivial for $m = 1$. Assume that it holds for all $m < n$, and let G be a simple graph which contains no K_{n+1}. Choose a vertex u of degree Δ in G, and set $G_1 = G[N(u)]$. Since G contains no K_{n+1}, G_1 contains no K_n and therefore, by the induction hypothesis, is degree-majorised by some complete $(n-1)$-partite graph H_1.

Next, set $V_1 = N(u)$ and $V_2 = V \backslash V_1$, and denote by G_2 the graph whose vertex set is V_2 and whose edge set is empty. Consider the join $G_1 \vee G_2$ of G_1 and G_2. Since

$$N_G(v) \subseteq N_{G_1 \vee G_2}(v) \quad \text{for} \quad v \in V_1 \tag{7.15}$$

and since each vertex of V_2 has degree Δ in $G_1 \vee G_2$, G is degree-majorised by $G_1 \vee G_2$. Therefore G is also degree-majorised by the complete n-partite graph $H = H_1 \vee G_2$. (See figure 7.3 for illustration.)

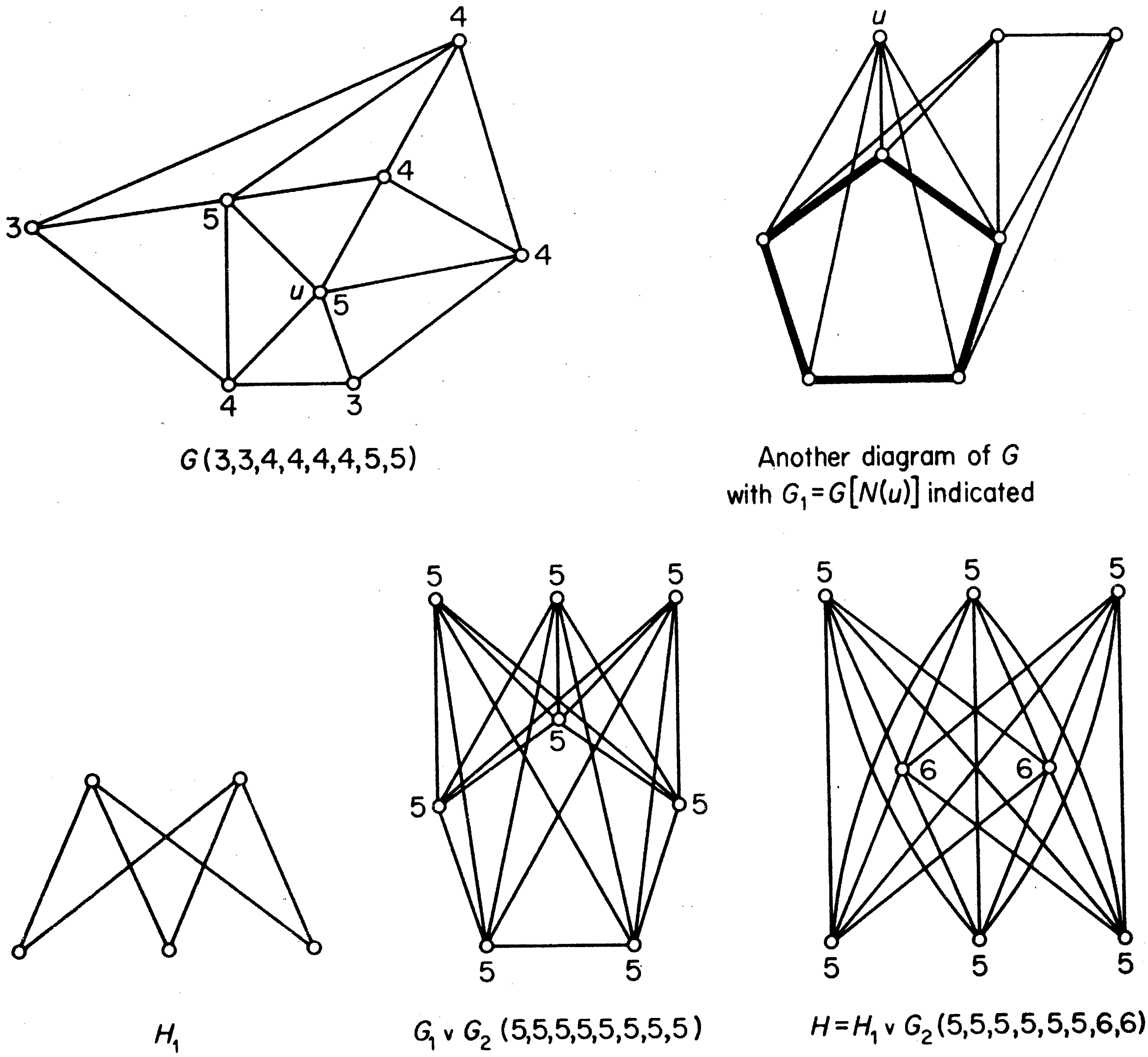


Figure 7.3

Suppose, now, that G has the same degree sequence as H. Then G has the same degree sequence as $G_1 \vee G_2$ and hence equality must hold in (7.15). Thus, in G, every vertex of V_1 must be joined to every vertex of V_2. It follows that $G = G_1 \vee G_2$. Since $G = G_1 \vee G_2$ has the same degree sequence as $H = H_1 \vee G_2$, the graphs G_1 and H_1 must have the same degree sequence and therefore, by the induction hypothesis, be isomorphic. We conclude that $G \cong H$ □

It is interesting to note that the above theorem bears a striking similarity to theorem 4.6.

Let $T_{m,n}$ denote the complete m-partite graph on n vertices in which all parts are as equal in size as possible; the graph H of figure 7.3 is $T_{3,8}$.

Theorem 7.9 If G is simple and contains no K_{m+1}, then $\varepsilon(G) \leq \varepsilon(T_{m,\nu})$. Moreover, $\varepsilon(G) = \varepsilon(T_{m,\nu})$ only if $G \cong T_{m,\nu}$.

Proof Let G be a simple graph that contains no K_{m+1}. By theorem 7.8, G is degree-majorised by some complete m-partite graph H. It follows from theorem 1.1 that

$$\varepsilon(G) \leq \varepsilon(H) \tag{7.16}$$

But (exercise 1.2.9)

$$\varepsilon(H) \leq \varepsilon(T_{m,\nu}) \tag{7.17}$$

Therefore, from (7.16) and (7.17)

$$\varepsilon(G) \leq \varepsilon(T_{m,\nu}) \tag{7.18}$$

proving the first assertion.

Suppose, now, that equality holds in (7.18). Then equality must hold in both (7.16) and (7.17). Since $\varepsilon(G) = \varepsilon(H)$ and G is degree-majorised by H, G must have the same degree sequence as H. Therefore, by theorem 7.8, $G \cong H$. Also, since $\varepsilon(H) = \varepsilon(T_{m,\nu})$, it follows (exercise 1.2.9) that $H \cong T_{m,\nu}$. We conclude that $G \cong T_{m,\nu}$ □

Exercises

7.3.1 In a group of nine people, one person knows two of the others, two people each know four others, four each know five others, and the remaining two each know six others. Show that there are three people who all know one another.

7.3.2 A certain bridge club has a special rule to the effect that four members may play together only if no two of them have previously partnered one another. At one meeting fourteen members, each of whom has previously partnered five others, turn up. Three games are played, and then proceedings come to a halt because of the club rule. Just as the members are preparing to leave, a new member, unknown to any of them, arrives. Show that at least one more game can now be played.

7.3.3 (*a*) Show that if G is simple and $\varepsilon > \nu^2/4$, then G contains a triangle.

(*b*) Find a simple graph G with $\varepsilon = [\nu^2/4]$ that contains no triangle.

(*c*)* Show that if G is simple and not bipartite with $\varepsilon > ((\nu-1)^2/4)+1$, then G contains a triangle.

(*d*) Find a simple non-bipartite graph G with $\varepsilon = [(\nu-1)^2/4]+1$ that contains no triangle. (P. Erdös)

7.3.4 (*a*)* Show that if G is simple and $\sum_{v \in V} \binom{d(v)}{2} > (m-1)\binom{\nu}{2}$, then G contains $K_{2,m} (m \geq 2)$.

(*b*) Deduce that if G is simple and $\varepsilon > \frac{(m-1)^{\frac{1}{2}}\nu^{\frac{3}{2}}}{2} + \frac{\nu}{4}$, then G contains $K_{2,m} (m \geq 2)$.

(c) Show that, given a set of n points in the plane, the number of pairs of points at distance exactly 1 is at most $n^{\frac{3}{2}}/\sqrt{2}+n/4$.

7.3.5 Show that if G is simple and $\varepsilon > \frac{(m-1)^{1/m}\nu^{2-1/m}}{2} + \frac{(m-1)\nu}{2}$ then G contains $K_{m,m}$.

APPLICATIONS

7.4 SCHUR'S THEOREM

Consider the partition $(\{1, 4, 10, 13\}, \{2, 3, 11, 12\}, \{5, 6, 7, 8, 9\})$ of the set of integers $\{1, 2, \ldots, 13\}$. We observe that in no subset of the partition are there integers x, y and z (not necessarily distinct) which satisfy the equation

$$x + y = z \tag{7.19}$$

Yet, no matter how we partition $\{1, 2, \ldots, 14\}$ into three subsets, there always exists a subset of the partition which contains a solution to (7.19). Schur (1916) proved that, in general, given any positive integer n, there exists an integer f_n such that, in any partition of $\{1, 2, \ldots, f_n\}$ into n subsets, there is a subset which contains a solution to (7.19). We shall show how Schur's theorem follows from the existence of the Ramsey numbers r_n (defined in exercise 7.2.3).

Theorem 7.10 Let $(S_1, S_2, \ldots, S_n)$ be any partition of the set of integers $\{1, 2, \ldots, r_n\}$. Then, for some i, S_i contains three integers x, y and z satisfying the equation $x + y = z$.

Proof Consider the complete graph whose vertex set is $\{1, 2, \ldots, r_n\}$. Colour the edges of this graph in colours $1, 2, \ldots, n$ by the rule that the edge uv is assigned colour j if and only if $|u - v| \in S_j$. By Ramsey's theorem (7.7) there exists a monochromatic triangle; that is, there are three vertices a, b and c such that ab, bc and ca have the same colour, say i. Assume, without loss of generality that $a > b > c$ and write $x = a - b$, $y = b - c$ and $z = a - c$. Then $x, y, z \in S_i$ and $x + y = z$ □

Let s_n denote the least integer such that, in any partition of $\{1, 2, \ldots, s_n\}$ into n subsets, there is a subset which contains a solution to (7.19). It can be easily seen that $s_1 = 2$, $s_2 = 5$ and $s_3 = 14$ (exercise 7.4.1). Also, from theorem 7.10 and exercise 7.2.3 we have the upper bound

$$s_n \le r_n \le [n!\,e] + 1$$

Exercise 7.4.2*b* provides a lower bound for s_n.

Exercises

7.4.1 Show that $s_1 = 2$, $s_2 = 5$ and $s_3 = 14$.

7.4.2 (*a*) Show that $s_n \geq 3s_{n-1} - 1$.

(*b*) Using (*a*) and the fact that $s_3 = 14$, show that $s_n \geq \frac{1}{2}(27(3)^{n-3} + 1)$. (A better lower bound has been obtained by Abbott and Moser, 1966.)

7.5 A GEOMETRY PROBLEM

The *diameter* of a set S of points in the plane is the maximum distance between two points of S. It should be noted that this is a purely geometric notion and is quite unrelated to the graph-theoretic concepts of diameter and distance.

We shall discuss sets of diameter 1. A set of n points determines $\binom{n}{2}$ distances between pairs of these points. It is intuitively clear that if n is 'large', then some of these distances must be 'small'. Therefore, for any d between 0 and 1, we can ask how many pairs of points in a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 can be at distance greater than d. Here, we shall present a solution to one special case of this problem, namely when $d = 1/\sqrt{2}$.

As an illustration, consider the case $n = 6$. We then have six points x_1, x_2, x_3, x_4, x_5 and x_6. If we place them at the vertices of a regular hexagon so that the pairs (x_1, x_4), (x_2, x_5) and (x_3, x_6) are at distance 1, as shown in figure 7.4*a*, these six points constitute a set of diameter 1.

It is easily calculated that the pairs (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_5, x_6) and (x_6, x_1) are at distance 1/2, and the pairs (x_1, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_6), (x_5, x_1) and (x_6, x_2) are at distance $\sqrt{3}/2$. Since $\sqrt{3}/2 > \sqrt{2}/2 = 1/\sqrt{2}$, there are nine pairs of points at distance greater than $1/\sqrt{2}$ in this set of diameter 1.

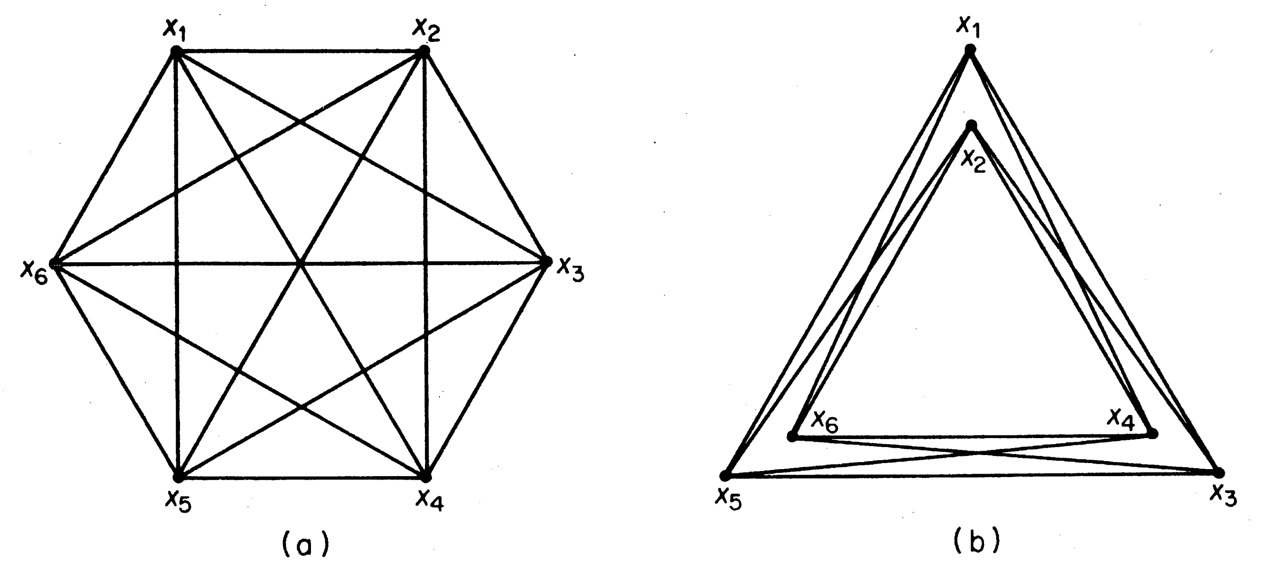


Figure 7.4

However, nine is not the best that we can do with six points. By placing the points in the configuration shown in figure 7.4*b*, all pairs of points except (x_1, x_2), (x_3, x_4) and (x_5, x_6) are at distance greater than $1/\sqrt{2}$. Thus we have twelve pairs at distance greater than $1/\sqrt{2}$; this is, in fact, the best we can do. The solution to the problem in general is given by the following theorem.

Theorem 7.11 If $\{x_1, x_2, \ldots, x_n\}$ is a set of diameter 1 in the plane, the maximum possible number of pairs of points at distance greater than $1/\sqrt{2}$ is $[n^2/3]$. Moreover, for each n, there is a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 with exactly $[n^2/3]$ pairs of points at distance greater than $1/\sqrt{2}$.

Proof Let G be the graph defined by

$$V(G) = \{x_1, x_2, \ldots, x_n\}$$

and

$$E(G) = \{x_i x_j \mid d(x_i, x_j) > 1/\sqrt{2}\}$$

where $d(x_i, x_j)$ here denotes the *euclidean* distance between x_i and x_j. We shall show that G cannot contain a K_4.

First, note that any four points in the plane must determine an angle of at least 90°. For the convex hull of the points is either (a) a line, (b) a triangle, or (c) a quadrilateral (see figure 7.5). Clearly, in each case there is an angle $x_i x_j x_k$ of at least 90°.

Now look at the three points x_i, x_j, x_k which determine this angle. Not all the distances $d(x_i, x_j)$, $d(x_i, x_k)$ and $d(x_j, x_k)$ can be greater than $1/\sqrt{2}$ and less than or equal to 1. For, if $d(x_i, x_j) > 1/\sqrt{2}$ and $d(x_j, x_k) > 1/\sqrt{2}$, then $d(x_i, x_k) > 1$. Since the set $\{x_1, x_2, \ldots, x_n\}$ is assumed to have diameter 1, it follows that, of any four points in G, at least one pair cannot be joined by an edge, and hence that G cannot contain a K_4. By Turán's theorem (7.9)

$$\varepsilon(G) \le \varepsilon(T_{3,n}) = [n^2/3]$$

One can construct a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 in which exactly

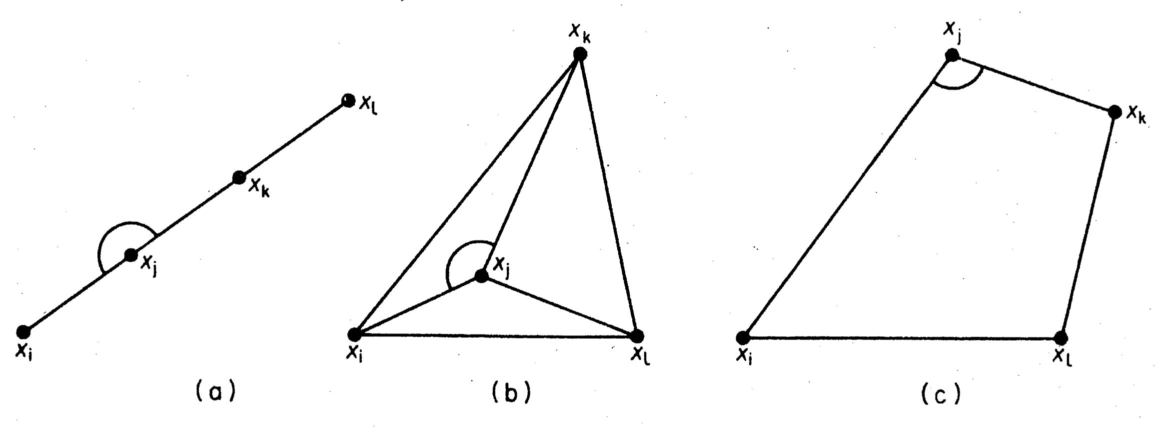


Figure 7.5

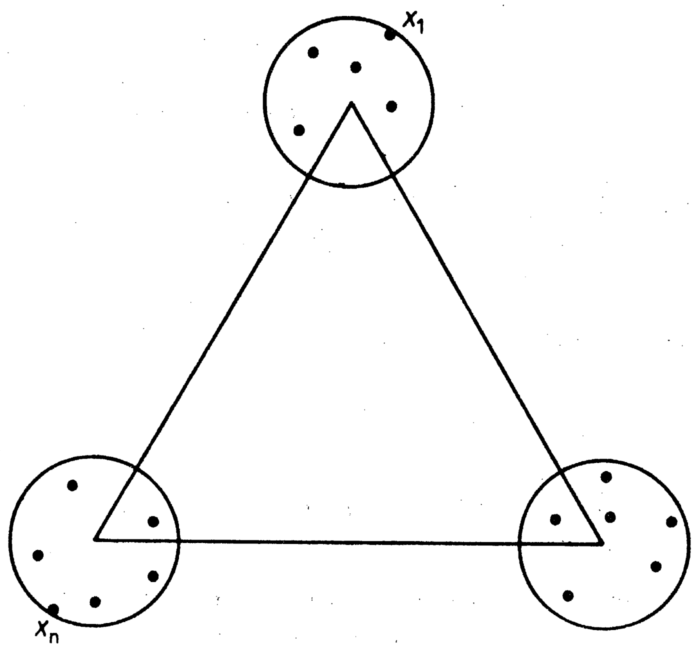


Figure 7.6

$[n^2/3]$ pairs of points are at distance greater than $1/\sqrt{2}$ as follows. Choose r such that $0<r<(1-1/\sqrt{2})/4$, and draw three circles of radius r whose centres are at a distance of $1-2r$ from one another (figure 7.6). Place $x_1, \ldots, x_{[n/3]}$ in one circle, $x_{[n/3]+1}, \ldots, x_{[2n/3]}$ in another, and $x_{[2n/3]+1}, \ldots, x_n$ in the third, in such a way that $d(x_1, x_n)=1$. This set clearly has diameter 1. Also, $d(x_i, x_j)>1/\sqrt{2}$ if and only if x_i and x_j are in different circles, and so there are exactly $[n^2/3]$ pairs (x_i, x_j) for which $d(x_i, x_j)>1/\sqrt{2}$ □

Exercises

7.5.1* Let $\{x_1, x_2, \ldots, x_n\}$ be a set of diameter 1 in the plane.

(*a*) Show that the maximum possible number of pairs of points at distance 1 is n.

(*b*) Construct a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 in the plane in which exactly n pairs of points are at distance 1. (E. Pannwitz)

7.5.2 A flat circular city of radius six miles is patrolled by eighteen police cars, which communicate with one another by radio. If the range of a radio is nine miles, show that, at any time, there are always at least two cars each of which can communicate with at least five other cars.

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8 Vertex Colourings

8.1 CHROMATIC NUMBER

In chapter 6 we studied edge colourings of graphs. We now turn our attention to the analogous concept of vertex colouring.

A *k-vertex colouring* of G is an assignment of k colours, $1, 2, \ldots, k$, to the vertices of G; the colouring is *proper* if no two distinct adjacent vertices have the same colour. Thus a proper k-vertex colouring of a loopless graph G is a partition $(V_1, V_2, \ldots, V_k)$ of V into k (possibly empty) independent sets. G is *k-vertex-colourable* if G has a proper k-vertex colouring. It will be convenient to refer to a 'proper vertex colouring' as, simply, a *colouring* and to a 'proper k-vertex colouring' as a *k-colouring;* we shall similarly abbreviate 'k-vertex-colourable' to *k-colourable.* Clearly, a graph is k-colourable if and only if its underlying simple graph is k-colourable. Therefore, in discussing colourings, we shall restrict ourselves to simple graphs; a simple graph is 1-colourable if and only if it is empty, and 2-colourable if and only if it is bipartite. The *chromatic number,* $\chi(G)$, of G is the minimum k for which G is k-colourable; if $\chi(G) = k$, G is said to be *k-chromatic.* A 3-chromatic graph is shown in figure 8.1. It has the indicated 3-colouring, and is not 2-colourable since it is not bipartite.

It is helpful, when dealing with colourings, to study the properties of a special class of graphs called critical graphs. We say that a graph G is *critical* if $\chi(H) < \chi(G)$ for every proper subgraph H of G. Such graphs were first investigated by Dirac (1952). A *k-critical* graph is one that is k-chromatic and critical; every k-chromatic graph has a k-critical subgraph. A 4-critical graph, due to Grötzsch (1958), is shown in figure 8.2.

An easy consequence of the definition is that every critical graph is connected. The following theorems establish some of the basic properties of critical graphs.

Theorem 8.1 If G is k-critical, then $\delta \geq k-1$.

Proof By contradiction. If possible, let G be a k-critical graph with $\delta < k-1$, and let v be a vertex of degree δ in G. Since G is k-critical, $G-v$ is $(k-1)$-colourable. Let $(V_1, V_2, \ldots, V_{k-1})$ be a $(k-1)$-colouring of $G-v$. By definition, v is adjacent in G to $\delta < k-1$ vertices, and therefore v must be nonadjacent in G to every vertex of some V_j. But then $(V_1, V_2, \ldots, V_j \cup \{v\}, \ldots, V_{k-1})$ is a $(k-1)$-colouring of G, a contradiction. Thus $\delta \geq k-1$ □

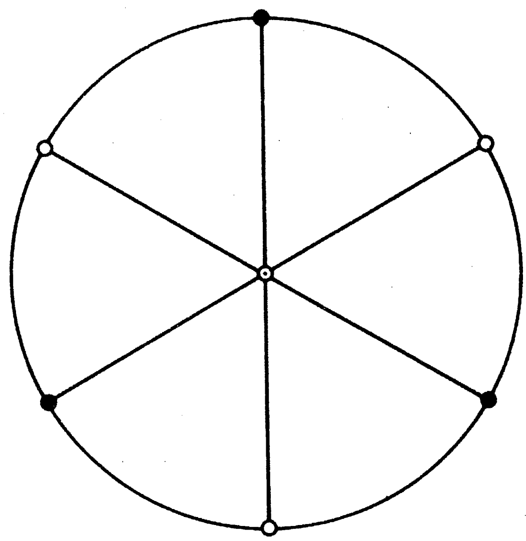

Figure 8.1. A 3-chromatic graph

Corollary 8.1.1 Every k-chromatic graph has at least k vertices of degree at least $k-1$.

Proof Let G be a k-chromatic graph, and let H be a k-critical subgraph of G. By theorem 8.1, each vertex of H has degree at least $k-1$ in H, and hence also in G. The corollary now follows since H, being k-chromatic, clearly has at least k vertices □

Corollary 8.1.2 For any graph G,

$$\chi \leq \Delta + 1$$

Proof This is an immediate consequence of corollary 8.1.1 □

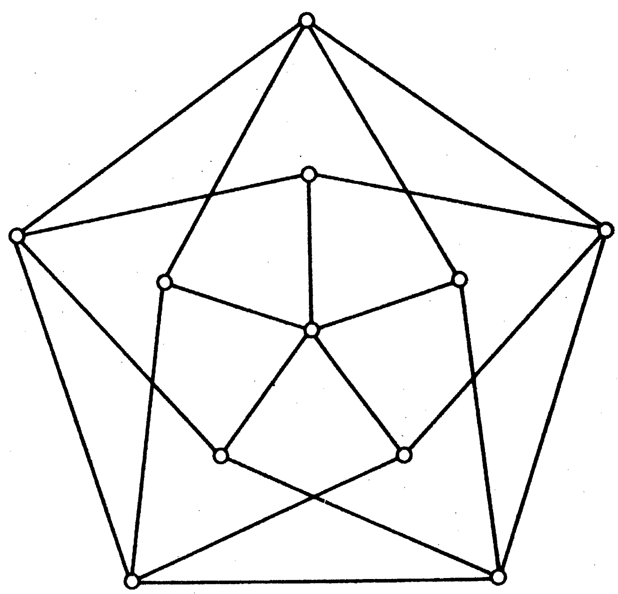

Figure 8.2. The Grötzsch graph—a 4-critical graph

Let S be a vertex cut of a connected graph G, and let the components of $G-S$ have vertex sets $V_1, V_2, \ldots, V_n$. Then the subgraphs $G_i = G[V \cup S]$ are called the S-*components* of G (see figure 8.3). We say that colourings of $G_1, G_2, \ldots, G_n$ *agree* on S if, for every $v \in S$, vertex v is assigned the same colour in each of the colourings.

Theorem 8.2 In a critical graph, no vertex cut is a clique.

Proof By contradiction. Let G be a k-critical graph, and suppose that G has a vertex cut S that is a clique. Denote the S-components of G b $G_1, G_2, \ldots, G_n$. Since G is k-critical, each G_i is $(k-1)$-colourable. Furth- more, because S is a clique, the vertices in S must receive distinct colours in any $(k-1)$-colouring of G_i. It follows that there are $(k-1)$-colourings of $G_1, G_2, \ldots, G_n$ which agree on S. But these colourings together yield a $(k-1)$-colouring of G, a contradiction □

Corollary 8.2 Every critical graph is a block.

Proof If v is a cut vertex, then $\{v\}$ is a vertex cut which is also, trivially, a clique. It follows from theorem 8.2 that no critical graph has a cut vertex; equivalently, every critical graph is a block □

Another consequence of theorem 8.2 is that if a k-critical graph G has a 2-vertex cut $\{u, v\}$, then u and v cannot be adjacent. We shall say that a $\{u, v\}$-component G_i of G is of *type 1* if every $(k-1)$-colouring of G_i assigns the same colour to u and v, and of *type 2* if every $(k-1)$-colouring of G_i assigns different colours to u and v (see figure 8.4).

Theorem 8.3 (Dirac, 1953) Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then

(i) $G = G_1 \cup G_2$, where G_i is a $\{u, v\}$-component of type i $(i = 1, 2)$, and

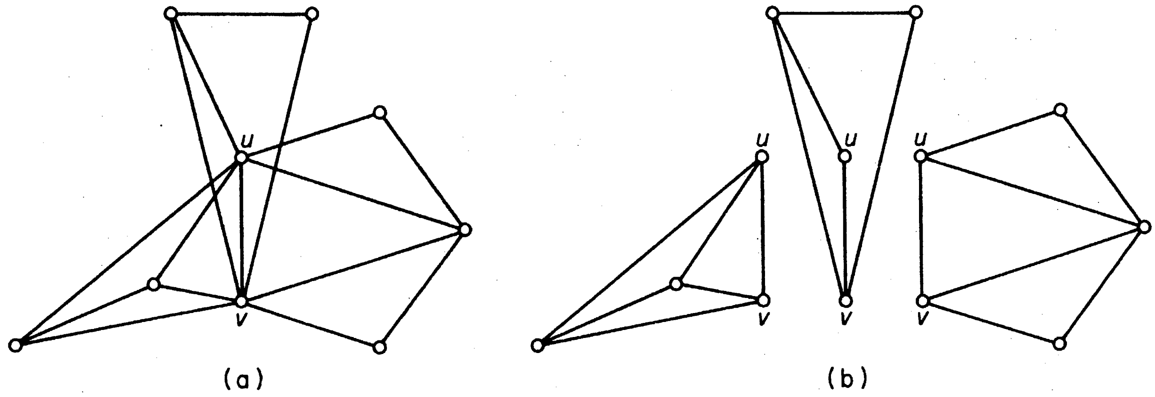

Figure 8.3. (*a*) G; (*b*) the $\{u, v\}$-components of G

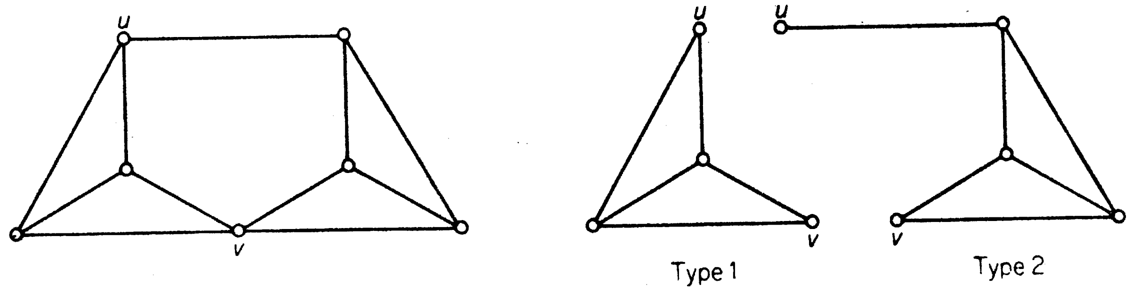

Figure 8.4

(ii) both G_1+uv and $G_2 \cdot uv$ are k-critical (where $G_2 \cdot uv$ denotes the graph obtained from G_2 by identifying u and v).,

Proof (i) Since G is critical, each $\{u, v\}$-component of G is $(k-1)$-colourable. Now there cannot exist $(k-1)$-colourings of these $\{u, v\}$-components all of which agree on $\{u, v\}$, since such colourings would together yield a $(k-1)$-colouring of G. Therefore there are two $\{u, v\}$-components G_1 and G_2 such that no $(k-1)$-colouring of G_1 agrees with any $(k-1)$-colouring of G_2. Clearly one, say G_1, must be of type 1 and the other, G_2, of type 2. Since G_1 and G_2 are of different types, the subgraph $G_1 \cup G_2$ of G is not $(k-1)$-colourable. Therefore, because G is critical, we must have $G = G_1 \cup G_2$.

(ii) Set $H_1 = G_1 + uv$. Since G_1 is of type 1, H_1 is k-chromatic. We shall prove that H_1 is critical by showing that, for every edge e of H_1, $H_1 - e$ is $(k-1)$-colourable. This is clearly so if $e = uv$, since then $H_1 - e = G_1$. Let e be some other edge of H_1. In any $(k-1)$-colouring of $G-e$, the vertices u and v must receive different colours, since G_2 is a subgraph of $G-e$. The restriction of such a colouring to the vertices of G_1 is a $(k-1)$-colouring of $H_1 - e$. Thus $G_1 + uv$ is k-critical. An analogous argument shows that $G_2 \cdot uv$ is k-critical □

Corollary 8.3 Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then

$$d(u)+d(v) \geq 3k-5 \tag{8.1}$$

Proof Let G_1 be the $\{u, v\}$-component of type 1 and G_2 the $\{u, v\}$-component of type 2. Set $H_1 = G_1 + uv$ and $H_2 = G_2 \cdot uv$. By theorems 8.3 and 8.1

$$d_{H_1}(u) + d_{H_1}(v) \geq 2k-2$$

and

$$d_{H_2}(w) \geq k-1$$

where w is the new vertex obtained by identifying u and v.
It follows that

$$d_{G_1}(u) + d_{G_1}(v) \geq 2k-4$$

and

$$d_{G_2}(u)+d_{G_2}(v)\geq k-1$$

These two inequalities yield (8.1) □

Exercises

8.1.1 Show that if G is simple, then $\chi\geq\nu^2/(\nu^2-2\varepsilon)$.

8.1.2 Show that if any two odd cycles of G have a vertex in common, then $\chi\leq 5$.

8.1.3 Show that if G has degree sequence $(d_1, d_2, \ldots, d_\nu)$ with $d_1\geq d_2\geq \ldots\geq d_\nu$, then $\chi\leq\max_i\min\{d_i+1, i\}$.

(D. J. A. Welsh and M. B. Powell)

8.1.4 Using exercise 8.1.3, show that

(*a*) $\chi\leq\{(2\varepsilon)^{\frac{1}{2}}\}$;

(*b*) $\chi(G)+\chi(G^c)\leq\nu+1$. (E. A. Nordhaus and J. W. Gaddum)

8.1.5 Show that $\chi(G)\leq 1+\max\delta(H)$, where the maximum is taken over all induced subgraphs H of G. (G. Szekeres and H. S. Wilf)

8.1.6* If a k-chromatic graph G has a colouring in which each colour is assigned to at least two vertices, show that G has a k-colouring of this type. (T. Gallai)

8.1.7 Show that the only 1-critical graph is K_1, the only 2-critical graph is K_2, and the only 3-critical graphs are the odd k-cycles with $k\geq 3$.

8.1.8 A graph G is *uniquely k-colourable* if any two k-colourings of G induce the same partition of V. Show that no vertex cut of a k-critical graph induces a uniquely $(k-1)$-colourable subgraph.

8.1.9 (*a*) Show that if u and v are two vertices of a critical graph G, then $N(u)\not\subseteq N(v)$.

(*b*) Deduce that no k-critical graph has exactly $k+1$ vertices.

8.1.10 Show that

(*a*) $\chi(G_1\vee G_2)=\chi(G_1)+\chi(G_2)$;

(*b*) $G_1\vee G_2$ is critical if and only if both G_1 and G_2 are critical.

8.1.11 Let G_1 and G_2 be two k-critical graphs with exactly one vertex v in common, and let vv_1 and vv_2 be edges of G_1 and G_2. Show that the graph $(G_1-vv_1)\cup(G_2-vv_2)+v_1v_2$ is k-critical. (G. Hajós)

8.1.12 For $n=4$ and all $n\geq 6$, construct a 4-critical graph on n vertices.

8.1.13 (*a*)* Let (X, Y) be a partition of V such that $G[X]$ and $G[Y]$ are both n-colourable. Show that, if the edge cut $[X, Y]$ has at most $n-1$ edges, then G is also n-colourable.

(P. C. Kainen)

(*b*) Deduce that every k-critical graph is $(k-1)$-edge-connected.

(G. A. Dirac)

8.2 BROOKS' THEOREM

The upper bound on chromatic number given in corollary 8.1.2 is sometimes very much greater than the actual value. For example, bipartite graphs are 2-chromatic, but can have arbitrarily large maximum degree. In this sense corollary 8.1.2 is a considerably weaker result than Vizing's theorem (6.2). There is another sense in which Vizing's result is stronger. Many graphs G satisfy $\chi' = \Delta + 1$ (see exercises 6.2.2 and 6.2.3). However, as is shown in the following theorem due to Brooks (1941), there are only two types of graph G for which $\chi = \Delta + 1$. The proof of Brooks' theorem given here is by Lovász (1973).

Theorem 8.4 If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Proof Let G be a k-chromatic graph which satisfies the hypothesis of the theorem. Without loss of generality, we may assume that G is k-critical. By corollary 8.2, G is a block. Also, since 1-critical and 2-critical graphs are complete and 3-critical graphs are odd cycles (exercise 8.1.7), we have $k \geq 4$. If G has a 2-vertex cut $\{u, v\}$, corollary 8.3 gives

$$2\Delta \geq d(u) + d(v) \geq 3k - 5 \geq 2k - 1$$

This implies that $\chi = k \leq \Delta$, since 2Δ is even.

Assume, then, that G is 3-connected. Since G is not complete, there are three vertices u, v and w in G such that $uv, vw \in E$ and $uw \notin E$ (exercise 1.6.14). Set $u = v_1$ and $w = v_2$ and let $v_3, v_4, \ldots, v_\nu = v$ be any ordering of the vertices of $G - \{u, w\}$ such that each v_i is adjacent to some v_j with $j > i$. (This can be achieved by arranging the vertices of $G - \{u, w\}$ in nonincreasing order of their distance from v.) We can now describe a Δ-colouring of G: assign colour 1 to $v_1 = u$ and $v_2 = w$; then successively colour $v_3, v_4, \ldots, v_\nu$, each with the first available colour in the list $1, 2, \ldots, \Delta$. By the construction of the sequence $v_1, v_2, \ldots, v_\nu$, each vertex v_i, $1 \leq i \leq \nu - 1$, is adjacent to some vertex v_j with $j > i$, and therefore to at most $\Delta - 1$ vertices v_j with $j < i$. It follows that, when its turn comes to be coloured, v_i is adjacent to at most $\Delta - 1$ colours, and thus that one of the colours $1, 2, \ldots, \Delta$ will be available. Finally, since v_ν is adjacent to two vertices of colour 1 (namely v_1 and v_2), it is adjacent to at most $\Delta - 2$ other colours and can be assigned one of the colours $2, 3, \ldots, \Delta$ □

Exercises

8.2.1 Show that Brooks' theorem is equivalent to the following statement: if G is k-critical ($k \geq 4$) and not complete, then $2\varepsilon \geq \nu(k-1)+1$.

8.2.2 Use Brooks' theorem to show that if G is loopless with $\Delta = 3$, then $\chi' \leq 4$.

8.3 HAJÓS' CONJECTURE

A *subdivision* of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions. A subdivision of K_4 is shown in figure 8.5. Although no necessary and sufficient condition for a graph to be k-chromatic is known when $k \geq 3$, a plausible necessary condition has been proposed by Hajós (1961): if G is k-chromatic, then G contains a subdivision of K_k. This is known as *Hajós' conjecture.* It should be noted that the condition is not sufficient; for example, a 4-cycle is a subdivision of K_3, but is not 3-chromatic.

For $k = 1$ and $k = 2$, the validity of Hajós' conjecture is obvious. It is also easily verified for $k = 3$, because a 3-chromatic graph necessarily contains an odd cycle, and every odd cycle is a subdivision of K_3. Dirac (1952) settled the case $k = 4$.

Theorem 8.5 If G is 4-chromatic, then G contains a subdivision of K_4.

Proof Let G be a 4-chromatic graph. Note that if some subgraph of G contains a subdivision of K_4, then so, too, does G. Without loss of generality, therefore, we may assume that G is critical, and hence that G is a block with $\delta \geq 3$. If $\nu = 4$, then G is K_4 and the theorem holds trivially. We proceed by induction on ν. Assume the theorem true for all 4-chromatic graphs with fewer than n vertices, and let $\nu(G) = n > 4$.

Suppose, first, that G has a 2-vertex cut $\{u, v\}$. By theorem 8.3, G has two $\{u, v\}$-components G_1 and G_2, where $G_1 + uv$ is 4-critical. Since $\nu(G_1 + uv) < \nu(G)$, we can apply the induction hypothesis and deduce that $G_1 + uv$

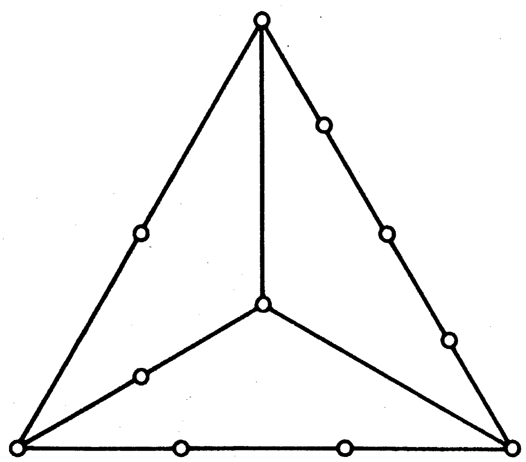

Figure 8.5. A subdivision of K_4

contains a subdivision of K_4. It follows that, if P is a (u, v)-path in G_2, then $G_1 \cup P$ contains a subdivision of K_4. Hence so, too, does G, since $G_1 \cup P \subseteq G$.

Now suppose that G is 3-connected. Since $\delta \geq 3$, G has a cycle C of length at least four. Let u and v be nonconsecutive vertices on C. Since $G-\{u, v\}$ is connected, there is a path P in $G-\{u, v\}$ connecting the two components of $C-\{u, v\}$; we may assume that the origin x and the terminus y are the only vertices of P on C. Similarly, there is a path Q in $G-\{x, y\}$ (see figure 8.6).

If P and Q have no vertex in common, then $C \cup P \cup Q$ is a subdivision of K_4 (figure 8.6*a*). Otherwise, let w be the first vertex of P on Q, and let P' denote the (x, w)-section of P. Then $C \cup P' \cup Q$ is a subdivision of K_4 (figure 8.6*b*). Hence, in both cases, G contains a subdivision of K_4 □

Hajós' conjecture has not yet been settled in general, and its resolution is known to be a very difficult problem. There is a related conjecture due to Hadwiger (1943): if G is k-chromatic, then G is 'contractible' to a graph which contains K_k. Wagner (1964) has shown that the case $k=5$ of Hadwiger's conjecture is equivalent to the famous four-colour conjecture, to be discussed in chapter 9.

Exercises

8.3.1* Show that if G is simple and has at most one vertex of degree less than three, then G contains a subdivision of K_4.

8.3.2 (*a*)* Show that if G is simple with $\nu \geq 4$ and $\varepsilon \geq 2\nu-2$, then G contains a subdivision of K_4.

(*b*) For $\nu \geq 4$, find a simple graph G with $\varepsilon = 2\nu-3$ that contains no subdivision of K_4.

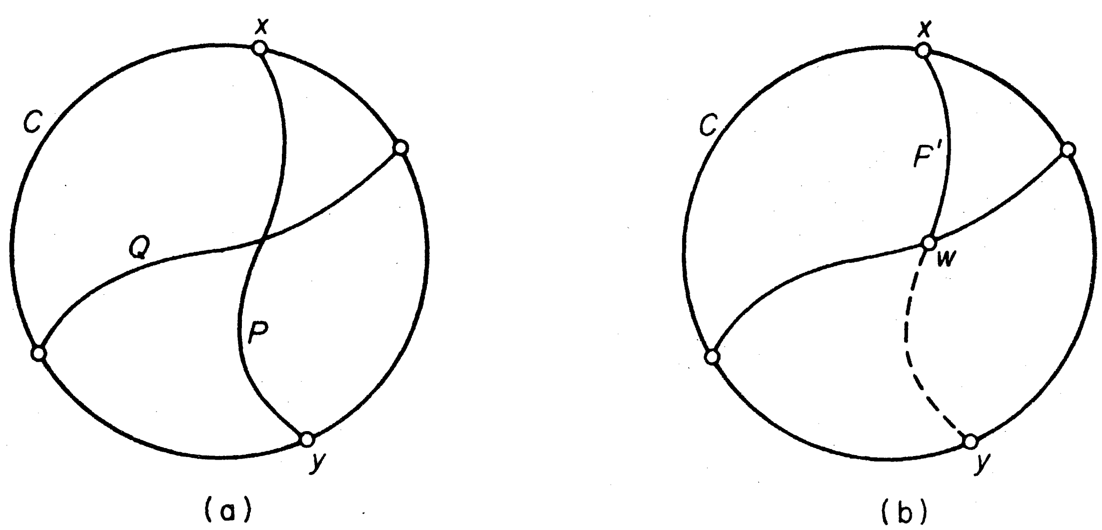

Figure 8.6

8.4 CHROMATIC POLYNOMIALS

In the study of colourings, some insight can be gained by considering not only the existence of colourings but the number of such colourings; this approach was developed by Birkhoff (1912) as a possible means of attacking the four-colour conjecture.

We shall denote the number of distinct k-colourings of G by $\pi_k(G)$; thus $\pi_k(G)>0$ if and only if G is k-colourable. Two colourings are to be regarded as distinct if some vertex is assigned different colours in the two colourings; in other words, if $(V_1, V_2, \ldots, V_k)$ and $(V_1', V_2', \ldots, V_k')$ are two colourings, then $(V_1, V_2, \ldots, V_k)=(V_1', V_2', \ldots, V_k')$ if and only if $V_i=V_i'$ for $1\leq i\leq k$. For example, a triangle has the six distinct 3-colourings shown in figure 8.7. Note that even though there is exactly one vertex of each colour in each colouring, we still regard these six colourings as distinct.

If G is empty, then each vertex can be independently assigned any one of the k available colours. Therefore $\pi_k(G)=k^\nu$. On the other hand, if G is complete, then there are k choices of colour for the first vertex, $k-1$ choices for the second, $k-2$ for the third, and so on. Thus, in this case, $\pi_k(G)=k(k-1)\ldots(k-\nu+1)$. In general, there is a simple recursion formula for $\pi_k(G)$. It bears a close resemblance to the recursion formula for $\tau(G)$ (the number of spanning trees of G), given in theorem 2.8.

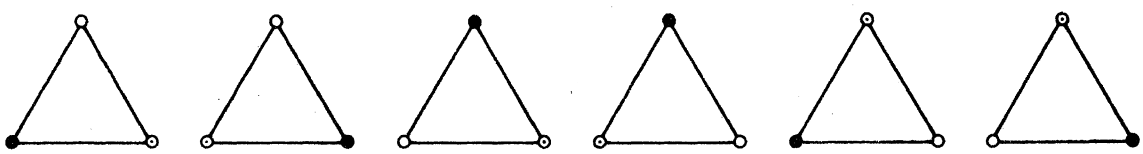

Figure 8.7

Theorem 8.6 If G is simple, then $\pi_k(G)=\pi_k(G-e)-\pi_k(G\cdot e)$ for any edge e of G.

Proof Let u and v be the ends of e. To each k-colouring of $G-e$ that assigns the same colour to u and v, there corresponds a k-colouring of $G\cdot e$ in which the vertex of $G\cdot e$ formed by identifying u and v is assigned the common colour of u and v. This correspondence is clearly a bijection (see figure 8.8). Therefore $\pi_k(G\cdot e)$ is precisely the number of k-colourings of $G-e$ in which u and v are assigned the same colour.

Also, since each k-colouring of $G-e$ that assigns different colours to u and v is a k-colouring of G, and conversely, $\pi_k(G)$ is the number of k-colourings of $G-e$ in which u and v are assigned different colours. It follows that $\pi_k(G-e)=\pi_k(G)+\pi_k(G\cdot e)$ □

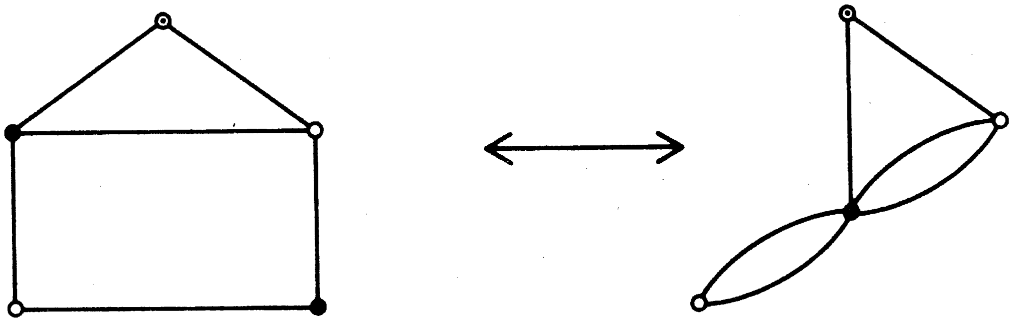

Figure 8.8

Corollary 8.6 For any graph G, $\pi_k(G)$ is a polynomial in k of degree ν, with integer coefficients, leading term k^ν and constant term zero. Furthermore, the coefficients of $\pi_k(G)$ alternate in sign.

Proof By induction on ε. We may assume, without loss of generality, that G is simple. If $\varepsilon = 0$ then, as has already been noted, $\pi_k(G) = k^\nu$, which trivially satisfies the conditions of the corollary. Suppose, now, that the corollary holds for all graphs with fewer than m edges, and let G be a graph with m edges, where $m \geq 1$. Let e be any edge of G. Then both $G-e$ and $G \cdot e$ have $m-1$ edges, and it follows from the induction hypothesis that there are non-negative integers $a_1, a_2, \ldots, a_{\nu-1}$ and $b_1, b_2, \ldots, b_{\nu-2}$ such that

$$\pi_k(G-e) = \sum_{i=1}^{\nu-1} (-1)^{\nu-i} a_i k^i + k^\nu$$

and

$$\pi_k(G \cdot e) = \sum_{i=1}^{\nu-2} (-1)^{\nu-i-1} b_i k^i + k^{\nu-1}$$

By theorem 8.6

$$\begin{aligned}\pi_k(G) &= \pi_k(G-e) - \pi_k(G \cdot e)\\ &= \sum_{i=1}^{\nu-2} (-1)^{\nu-i}(a_i + b_i)k^i - (a_{\nu-1}+1)k^{\nu-1} + k^\nu\end{aligned}$$

Thus G, too, satisfies the conditions of the corollary. The result follows by the principle of induction □

By virtue of corollary 8.6, we can now refer to the function $\pi_k(G)$ as the *chromatic polynomial* of G. Theorem 8.6 provides a means of calculating the chromatic polynomial of a graph recursively. It can be used in either of two ways:

(i) by repeatedly applying the recursion $\pi_k(G) = \pi_k(G-e) - \pi_k(G \cdot e)$, and thereby expressing $\pi_k(G)$ as a linear combination of chromatic polynomials of empty graphs, or
(ii) by repeatedly applying the recursion $\pi_k(G-e) = \pi_k(G) + \pi_k(G \cdot e)$, and

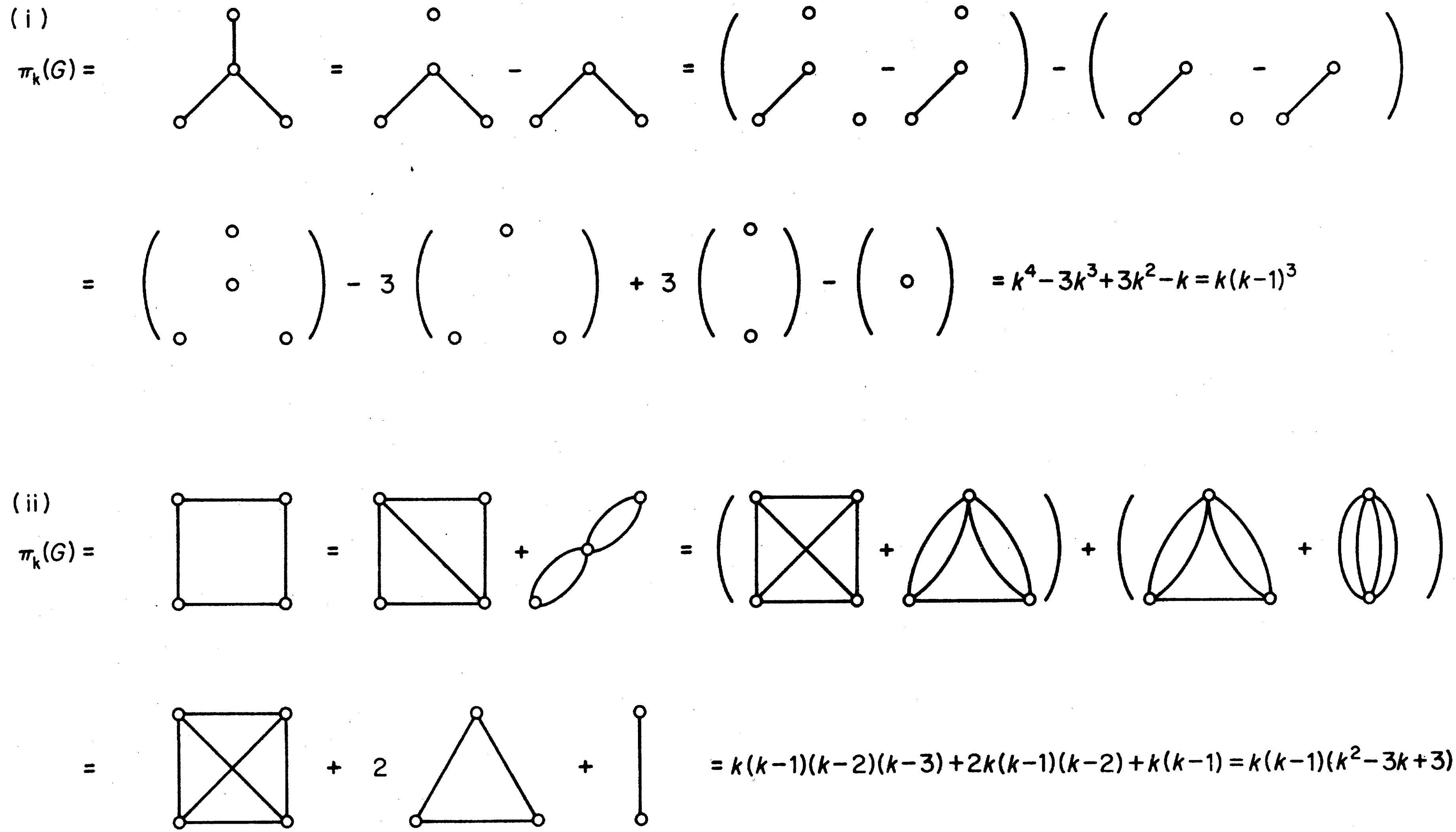

Figure 8.9. Recursive calculation of $\pi_k(G)$