DISCRETE MATHEMATICS 10 (1974) 383–390. © North-Holland Publishing Company

ON HYPOHAMILTONIAN GRAPHS

Carsten THOMASSEN*
Mathematisk Institut, Universitetsparken, Ny Munkegade, 8000 Aarhus C, Denmark

Received 22 January 1974

Abstract. Herz, Duby and Vigué [9] conjectured that every hypohamiltonian graph has girth $\geqslant 5$. In the present note hypohamiltonian graphs of girth 3 and 4 are described. Also two conjectures on hypohamiltonian graphs made by Bondy and Chvátal, respectively, are disproved.

1. Introduction and terminology

We adopt the notation and terminology of Harary [8] with the modifications that the terms *vertices* and *edges* are here used instead of the terms *points* and *lines*, respectively, in [8]. The set of vertices, respectively edges, of the graph G is denoted by $V(G)$, respectively $E(G)$. The edge joining the vertices x and y is denoted by (x, y) and (y, x) and the degree of x in G is denoted by $d(x, G)$.

A graph G is *hypohamiltonian* if and only if G is not Hamiltonian but every vertex-deleted subgraph $G - v$ is Hamiltonian. Hypohamiltonian graphs were first studied by Sousselier (see [1, 2]) who among other things proved that the Petersen graph is the smallest one. Herz, Duby and Vigué [9] proved that there exists no hypohamiltonian graph with 11 or 12 vertices. Infinite families of hypohamiltonian graphs have been constructed by Sousselier (see [9]), Lindgren [11], Bondy [3], Chvátal [4], Doyen and Van Diest [7] and by the author [12]. In [12] it was shown that for every $p \geqslant 13$, except possibly for $p = 14, 17, 19$, there exists a hypohamiltonian graph with p vertices. This improved on the result of Chvátal [4] for $p = 20, 25$. Doyen and Van Diest have constructed hypohamiltonian graphs with $3k + 1$ vertices for all $k \geqslant 3$ so the question of the existence of a hypohamiltonian graph with p vertices is left open for $p = 14, 17$.

* Present address: Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ont., Canada.

The following three conjectures have been made concerning the structure of hypohamiltonian graphs.

(1) *Every hypohamiltonian graph has girth* ≥ 5. (Herz, Duby and Vigué [9]. See also [4, 5, 6].)

(2) *If the deletion of an edge e from a hypohamiltonian graph G does not create a vertex of degree two, then* $G - e$ *is hypohamiltonian* (Chvátal [4]).

(3) *If the addition of a new edge to a hypohamiltonian graph of girth* ≥ 5 *does not create a cycle of length* < 5, *then it does not create a Hamiltonian cycle* (Bondy, see [4]).

This note gives examples of hypohamiltonian graphs for which (1) and (2) are false and one for which (2) and (3) are false.

2. Construction of hypohamiltonian graphs

Let G_1, G_2 be disjoint graphs. Assume G_1, respectively G_2, contains a vertex x_0, respectively y_0, of degree 3, and let x_1, x_2, x_3, respectively y_1, y_2, y_3, denote the vertices adjacent to x_0, respectively y_0. Assume that G_2 is hypohamiltonian. Bondy (see [4, p. 39]) pointed out that G_2 contains none of the edges (y_1, y_2), (y_1, y_3), (y_2, y_3). We assume that the graph G_1 has at least six vertices. Let G denote the graph obtained from $H_1 = G_1 - x_0$ and $H_2 = G_2 - y_0$ by identifying the vertices x_1, y_1 into a vertex z_1, the vertices x_2, y_2 into a vertex z_2 and the vertices x_3, y_3 into a vertex z_3. This construction is illustrated in [12, Fig. 1]. The special case in which G_2 is the Petersen graph is shown in Fig. 1. In this case we say that x_0 *is replaced by a vertex-deleted subgraph of the Petersen graph*. We consider H_1 and H_2 as subgraphs of G. In [12] it was shown that G is hypohamiltonian provided G_1 is hypohamiltonian. By the same type of arguments we obtain the following stronger result.

Lemma 1. (a) *G is Hamiltonian if and only if* G_1 *is Hamiltonian.*

(b) *For every* $z \in V(H_1)$, $G - z$ *is Hamiltonian if and only if* $G_1 - z$ *is Hamiltonian.*

(c) *If* $G_1 - x_i$ *is Hamiltonian for* $i = 1, 2, 3$, *then for every* $z \in V(H_2)$, $G - z$ *is Hamiltonian.*

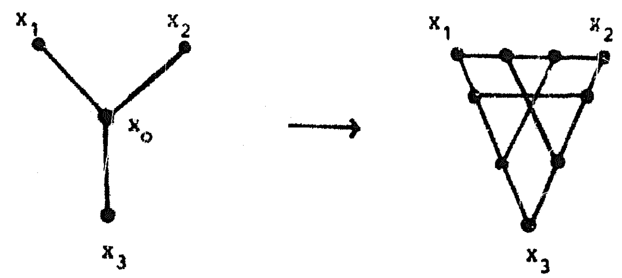

Fig. 1. Replacement of x_0 by a vertex-deleted subgraph of the Petersen graph.

Proof. Suppose first that G_1 is Hamiltonian. Let C be a Hamiltonian cycle of G_1. Then $P_1 = C - x_0$ is a Hamiltonian path of H_1 connecting two of the vertices x_1, x_2, x_3 (x_1 and x_2, say). Since $G_2 - y_3$ is Hamiltonian, $G_2 - y_3 - y_0 = H_2 - y_3$ contains a Hamiltonian path P_2 connecting y_1 and y_2. Then $P_1 \cup P_2$ is a Hamiltonian cycle of G. Suppose next that G is Hamiltonian and let C be a Hamiltonian cycle of G. $C = P_1 \cup P_2 \cup P_3$, where P_1 is a $z_1 - z_2$ path, P_2 is a $z_2 - z_3$ path and P_3 is a $z_3 - z_1$ path. Each of the paths P_i is a path of either H_1 or H_2 for $i = 1, 2, 3$. Two of these paths are contained in H_j, where $j = 1$ or $j = 2$. Then H_j has a Hamiltonian path connecting two of the vertices z_1, z_2, z_3 and clearly G_j is Hamiltonian. Since G_2 is assumed to be non-Hamiltonian, we have proved that G_1 is Hamiltonian and we have proved (a). If $z \in V(H_1) - \{z_1, z_2, z_3\}$, then, by (a), $G - z$ is Hamiltonian if and only if $G_1 - z$ is Hamiltonian since $G - z$ is obtained from $G_1 - z$ and G_2 in the same way as G is obtained from G_1 and G_2. Since $H_2 - y_i$ ($i = 1, 2, 3$) has a Hamiltonian path connecting the two vertices of $\{y_1, y_2, y_3\} - \{y_i\}$, clearly $G - z_i$ is Hamiltonian if and only if $G_1 - x_i$ is Hamiltonian. This proves (b). If $z \in V(H_2)$, then $H_2 - z$ has a Hamiltonian path P_2 connecting two of the vertices y_1, y_2, y_3 (y_1 and y_2, say). If $G_1 - x_3$ is Hamiltonian, then $H_1 - x_3$ contains a Hamiltonian path P_1 connecting x_1 and x_2 and $P_1 \cup P_2$ is a Hamiltonian cycle of $G - z$, so (c) holds.

Theorem 1. *Let G be a non-Hamiltonian graph and let $A \subseteq V(G)$. Suppose that the vertices of A are mutually non-adjacent and that they all have degree 3. If for every vertex $z \in V(G) - A$, $G - z$ is Hamiltonian, then there exists a hypohamiltonian graph G' containing $G - A$ as a subgraph. If furthermore for every edge $e \in E(G - A)$ there is a vertex $z_e \in V(G) - A$ such that $G - e - z_e$ is non-Hamiltonian, then we can construct G' such that for every edge $e \in E(G')$, $G' - e$ is not hypohamiltonian.*

Proof. Let x_0 be any vertex of A. Replace x_0 by a vertex-deleted subgraph of the Petersen graph. Denote the resulting graph by G_1 and put $A_1 = A - \{x_0\}$. Then for every vertex $z \in V(G_1) - A_1$, $G_1 - z$ is Hamiltonian by Lemma 1. The vertices of A_1 are mutually non-adjacent and they all have degree 3 in G_1. If $e \in E(G - A)$, $z_e \in V(G) - A$ and $G - e - z_e$ is non-Hamiltonian, then also $G_1 - e - z_e$ is non-Hamiltonian by Lemma 1. If e is any edge of G_1 not contained in G, then $G_1 - e$ contains a vertex of degree 2. So it is easy to see that G_1 contains a vertex $z_e \in V(G_1) - A_1$ such that $G_1 - e - z_e$ is non-Hamiltonian. If $A_1 = \emptyset$, G_1 has the desired properties. If $A_1 \neq \emptyset$, we replace any vertex x_1 of A_1 by a vertex-deleted subgraph of the Petersen graph and we put $A_2 = A_1 - \{x_1\}$, etc. Since $|A| > |A_1| > |A_2| > \ldots$, we obtain in a finite number of steps a graph G' which satisfies the assertion of the theorem.

3. Disproof of the conjectures (1), (2), (3)

Using Theorem 1, it is easy to see that there exists a hypohamiltonian graph containing a cycle of length 4. Let for $k \geq 2$, R_k denote the graph consisting of the vertices

$$\{x_1, x_2, \ldots, x_{2k+1}, y_1, y_2, \ldots, y_{2k+1}, z_1, z_2\}$$

and the edges

$$\{(x_1, x_2), (x_2, x_3), \ldots, (x_{2k}, x_{2k+1}), (x_{2k+1}, x_1), (y_1, y_2), (y_2, y_3), \ldots,$$

$$(y_{2k+1}, y_1), (x_1, z_1), (z_1, y_1), (x_2, y_2), (x_3, z_2), (z_2, y_3), (x_4, y_4),$$

$$(x_5, y_5), \ldots, (x_{2k+1}, y_{2k+1})\} \,.$$

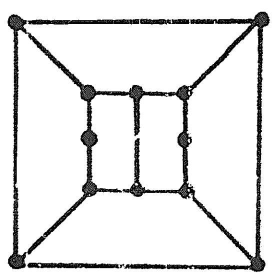

Fig. 2. The graph R_2.

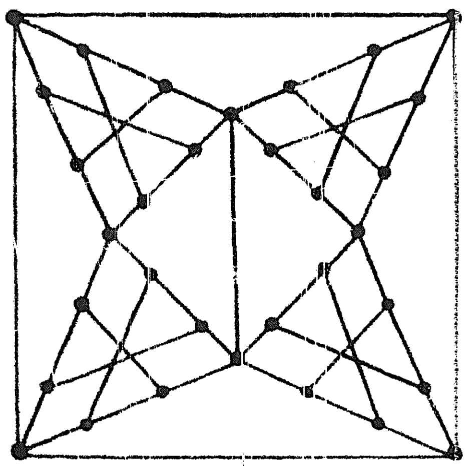

Fig. 3. A hypohamiltonian graph of girth 4.

R_2 is obtained from a pentagonal prism by subdividing two edges through the insertion of two new vertices of degree 2 (see Fig. 2). Put $A_k = \{x_1, y_1, x_3, y_3\} \subseteq V(R_k)$. Tutte [13] pointed out that R_2 is non-Hamiltonian. More generally, it is easy to see that R_k is non-Hamiltonian for all $k \geqslant 2$ and that $R_k - z$ is Hamiltonian whenever $z \in V(R_k) - A_k$. Also it is easy to see that for every edge $e \in E(R_k - A_k)$ there is a $z_e \in V(R_k) - A_k$ such that $R_k - e - z_e$ is non-Hamiltonian. By Theorem 1, there exists a hypohamiltonian graph $R'_k \supseteq R_k - A_k$ such that for any edge $e \in E(R'_k)$, $R'_k - e$ is not hypohamiltonian. R'_2 is shown in Fig. 3.

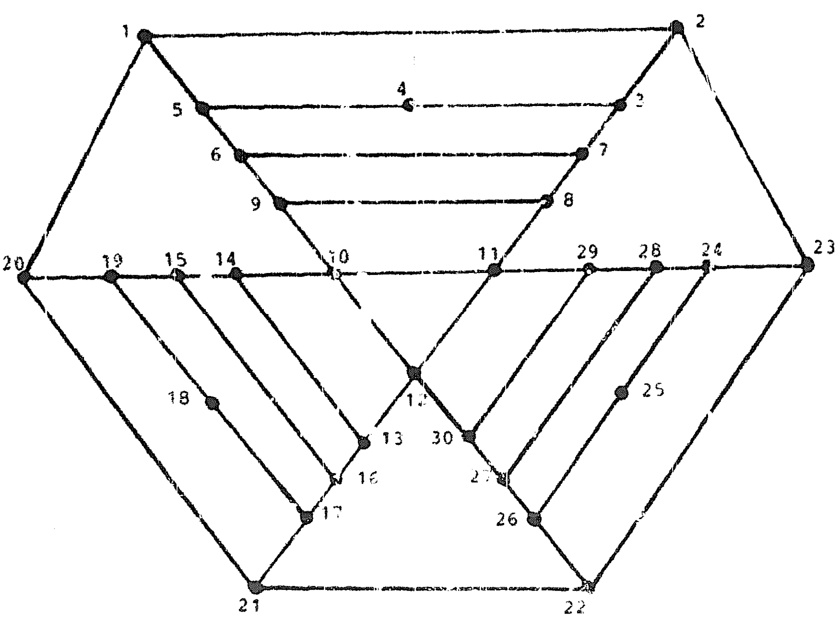

Fig. 4. The graph M.

Clearly, R'_k contains a cycle of length 4 and it contains edges whose removal does not create vertices of degree 2. So for every $k \geqslant 2$, R'_k is a counterexample to (1) and (2).

We shall go a step further and show that a hypohamiltonian graph may contain a cycle of length 3. Let M denote the graph in Fig. 4. $V(M) = \{1, 2, ..., 30\}$. Put $A = \{3, 5, 17, 19, 24, 26\}$. We shall show by reductio ad absurdum that M is non-Hamiltonian. Suppose C is a Hamiltonian cycle of M. Then C contains the edges (3, 4), (4, 5), (17, 18), (18, 19), (24, 25), (25, 26). Suppose first that C contains the edges (1, 5), (3, 7). Then C contains the edges (1, 2), (2, 23), (7, 6), (6, 9), (9, 8), (8, 11). Also C contains (21, 20), (20, 19), (17, 16), (16, 15), (15, 14), (14, 13), (13, 12). The two edges of C which are incident with 10 are then (10, 12) and (10, 11). C must contain the edge (21, 22) and if C contains (22, 23) also then C contains a cycle as a proper subgraph. So C does not contain (22, 23). But then C contains (22, 26) and (23, 24) and again we see that C contains a cycle as a proper subgraph, which is a contradiction. By symmetry, C cannot contain the edges (2, 3), (5, 6). So C contains either none of both of the edges (1, 5), (2, 3), or, in other words, C either contains the path 20, 1, 2, 23 or the path 20, 1, 5, 4, 3, 2, 23. Because of the symmetry, C contains either none of both of the edges (19, 20), (17, 21) and either none of both of the edges (22, 26), (23, 24). It is, however, easy to see that this leads to a contradiction and we have proved that M is non-Hamiltonian.

Next we show that $M - z$ is Hamiltonian whenever $z \in V(M) - A$. Because of the symmetry, it is sufficient to consider the cases $z = 1, 4, 6, 8, 10$. In the case $z = 4$, $M - z$ has the following Hamiltonian cycle: 1, 5, 6, 9, 8, 7, 3, 2, 23, 22, 26, 25, 24, 28, 27, 30, 29, 11, 12, 10, 14, 13, 16, 15, 19, 18, 17, 21, 20, 1. Let P denote the path 11, 29, 28, 24, 25, 26, 27, 30, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 2 and let P_1, P_6, P_8, P_{10} be the paths defined as follows:

P_1 : 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,
P_6 : 2, 1, 5, 4, 3, 7, 8, 9, 10, 11,
P_8 : 2, 1, 5, 4, 3, 7, 6, 9, 10, 11,
P_{10}: 2, 1, 5, 4, 3, 7, 6, 9, 8, 11.

Then $P \cup P_z$ is a Hamiltonian cycle of $M - z$ for $z = 1, 6, 8, 10$. Furthermore we can show that for every edge $e \in E(M - A)$ there exists a $z_e \in V(M) - A$ such that $M - e - z_e$ is non-Hamiltonian. If e is incident with a vertex of degree 3, then this vertex in $M - e$ is adjacent to a vertex $z_e \notin A$. Clearly, $M - e - z_e$ is non-Hamiltonian. If, on the other hand, e joins two vertices of degree $\geqslant 4$, then e is one of the edges (10, 11),

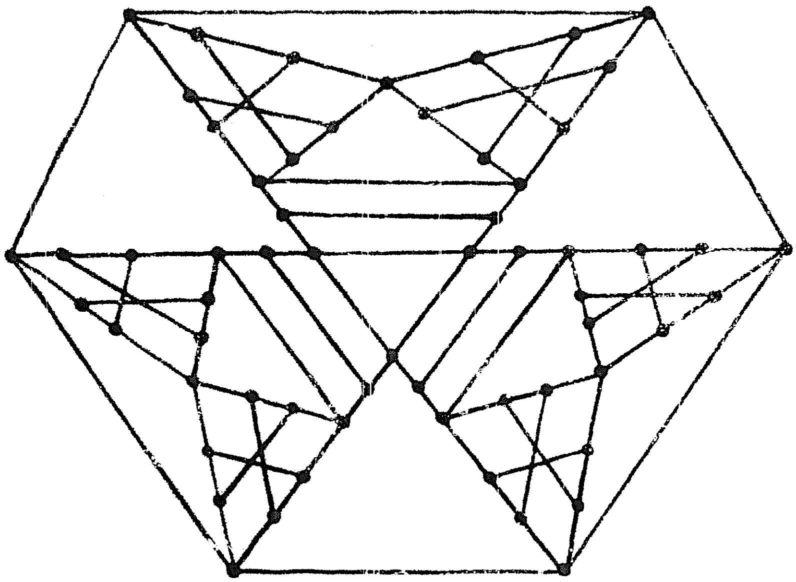

Fig. 5. A hypohamiltonian graph of girth 3.

(11, 12), (12, 10). If $e = (10, 11)$, we put $z_e = 1$ and it is easy to prove that $M - e - z_e$ is non-Hamiltonian (we leave this to the reader). By Theorem 1 there exists a hypohamiltonian graph M' (Fig. 5) such that M' contains $M - A$ as a subgraph and for any edge e of M', $M' - e$ is not hypohamiltonian. Clearly, M' is another counterexample to the conjectures (1) and (2).

We shall finally give a counterexample to the conjectures (2) and (3). Let G denote the Petersen graph and let A be a set consisting of two non-adjacent vertices of G. For every $z \in V(G) - A$, $G - z$ is Hamiltonian and for every $e \in E(G - A)$ there exists a $z_e \in V(G) - A$ such that $G - e - z_e$ is non-Hamiltonian. Let G' denote the graph obtained from G by replacing each vertex of A by a vertex-deleted subgraph of the Petersen graph (Fig. 6). Then G' is hypohamiltonian and the deletion of

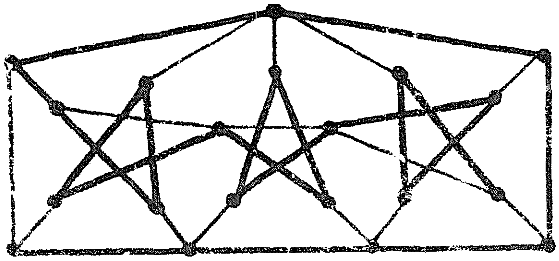

Fig. 6. A counterexample to conjectures (2), (3).

any edge of G' results in a graph which is not hypohamiltonian. So G' is clearly a counterexample to (2). A Hamiltonian path of G' is drawn with thick lines in Fig. 6. If we add the edge joining the endvertices of this path we create a Hamiltonian cycle of G' but we do not create a cycle of length < 5. So G' is a counterexample to conjecture (3).

References

[1] C. Berge, Problèmes plaisants et delectables No. 29, Rev. Française Informat. Recherche Opérationnelle 7 (1963) 405–406.

[2] C. Berge, Graphes et hypergraphes (Dunod, Paris, 1970).

[3] J.A. Bondy, Variations on the hamiltonian theme, Can. Math. Bull. 15 (1972) 57–62.

[4] V. Chvátal, Flip-flops in hypohamiltonian graphs, Can. Math. Bull. 16 (1973) 33–41.

[5] V. Chvátal, New directions in hamiltonian graph theory, in: F. Harary, ed., New Directions in Graph Theory (Academic Press, New York, 1973) 81.

[6] V. Chvátal, D.A. Klarner and D.E. Knuth, Selected combinatorial research problems, Stanford University, Stanford (1972) Problem 19.

[7] J. Doyen and V. Van Diest, New families of hypohamiltonian graphs, to appear.

[8] F. Harary, Graph Theory (Addison-Wesley, Reading, Mass., 1969).

[9] J.C. Herz, J.J. Duby and F. Vigué, Recherche systématique des graphes hypohamiltonien, in: P. Rosentiehl, ed., Theorie des graphes (Dunod, Paris, 1967) 153–160.

[10] J.C. Herz, T. Gaudin and P. Rossi, Solution du Problème No. 29, Rev. Française Informat. Recherche Opérationnelle 8 (1964) 214–218.

[11] W.F. Lindgren, An infinite class of hypohamiltonian graphs, Am. Math. Monthly 74 (1967) 1087–1089.

[12] C. Thomassen, Hypohamiltonian and hypotraceable graphs, Discrete Math. 9 (1974) 91–96.

[13] W.T. Tutte, On hamiltonian circuits, J. London Math. Soc. 21 (1947) 98–101.