2. Location Problems

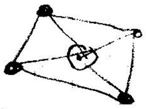
21.04.12

2.1 Introduction

2.1.1 Motivation: given a set V of points (in $\mathbb{R}^d$, in network N) find a number of medians (in $\mathbb{R}^d$, in N), i.e., new points, that minimize (sum, max) the distance ($l_2, l_2^2, l_1, l_\infty$) to V.

a) Median point in the plane (Fermat [17th century]): given a triangle, find a median (point in the plane) that min. the sum of the dist. to the Δ-vertices.

b) Location-allocation problem (Weber [20th cent.]): As Fermat, but $n \geq 3$ points, $p \geq 1$ facilities (median), distance weights w_p to account for customer demands



c) Absolute median problem (Hakimi [1960s]): $V \mathrel{\hat{=}}$ vertices of a graph, medians $\mathrel{\hat{=}}$ points on vertices and edges, dist $\mathrel{\hat{=}}$ dist in graph.

2.1.2 Def. (Classification Scheme for Location Problems, Hamacher & Nickel [1998]):

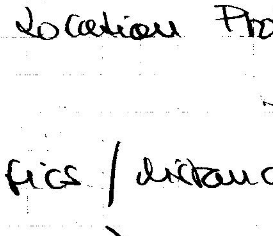
Input: new locations / domain / specifics / distances / objective

- new locations: $p = 1$ (one), N (many)
- domain: $\mathbb{R}^d$, $\mathbb{R}^2$ (planar), N (network), D (discrete $\mathrel{\hat{=}}$ finite set)
- specifics: R = ... (restricted positions), B = ... (barrier)
- distances: l_2, l_2^2, l_1, l_∞
- objective: Σ (of distances), max (of dist.)
- given points: finite $V \subseteq$ domain (implicit)

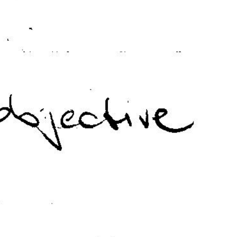
Output: P = argmin obj, $P \subseteq$ domain, specifics

2.1.3 Ex. (Schöbel & Schmidt [2008]):

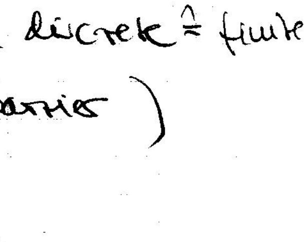
a) Desert well: $1/\mathbb{R}^2/\cdot/l_2/\Sigma$



b) Manhattan fire brigade: $1/\mathbb{R}^2/R=\text{buildings}/l_1/\max$

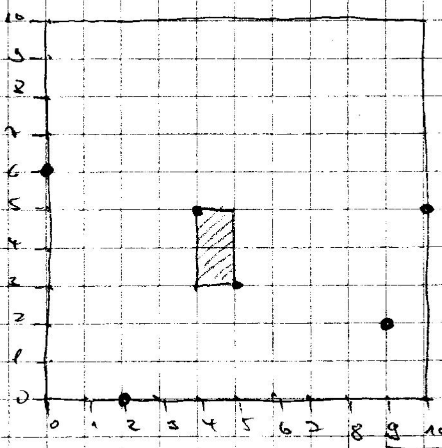


c) Warehouses: $N/N/\cdot/\text{shortest path}/\Sigma$



22. Medians in the Plane

2.2.1 Ex. (Manhattan distances, Schöbel & Schmidt [2009]):

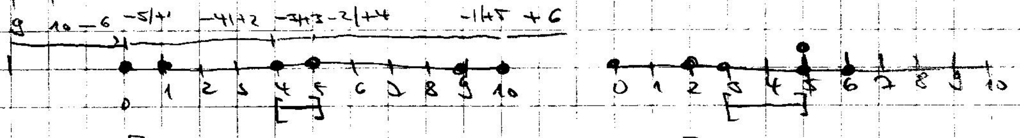


$$V = \{(a_i,b_i)\}_{i=1}^{6} = \{(0,6),(4,5),(10,5),(5,3),(9,2),(2,0)\}$$

$1/\mathbb{R}^2/\cdot/l_1/\Sigma$

(x,y)

$$\min \sum_{i=1}^{6} \left(|x-a_i| + |y-b_i|\right)$$
$$= \min \sum_{i=1}^{6} |x-a_i| + \min \sum_{i=1}^{6} |y-b_i| \quad \text{(separable)}$$



$$= \left[\operatorname{argmin}_3 \{a_i\}, \operatorname{argmin}_4 \{a_i\}\right] \times \left[\operatorname{argmin}_3 \{b_i\}, \operatorname{argmin}_4 \{b_i\}\right]$$
$$= \operatorname{med}\{a_i\} \times \operatorname{med}\{b_i\}.$$

2.2.2 Def. (Median of numbers): Let $\{x_1,\dots,x_m\} \subseteq \mathbb{R}$ be a set of $m \in \mathbb{N}$ numbers. Then
$$\operatorname{med}\{x_i\}_{i=1}^{m} := \begin{cases} \operatorname{argmin}_{\lceil \frac{m}{2} \rceil}\{x_i\}, & m \text{ odd} \\ \left[\operatorname{argmin}_{\frac{m}{2}}\{x_i\}, \operatorname{argmin}_{\frac{m}{2}+1}\{x_i\}\right], & m \text{ even} \end{cases}$$

2.2.3 Thm (explicit solution formula for $1/\mathbb{R}^2/\cdot/l_1/\Sigma$): Let $V=\{(a_i,b_i)\}_{i=1}^{m}$.
$$\operatorname*{argmin}_{(x,y)\in\mathbb{R}^2} 1/\mathbb{R}^2/\cdot/l_1/\Sigma w_i = \operatorname{med}\{a_i\}_{i=1}^{m} \times \operatorname{med}\{b_i\}_{i=1}^{m}. \quad \square$$

2.2.4 Ex (l_2^2): $V=\{(a_i,b_i)\}_{i=1}^{m}$, $1/\mathbb{R}^2/\cdot/l_2^2/\Sigma w_i$:
$$\min \sum_{i=1}^{m} w_i\left[(x-a_i)^2 + (y-b_i)^2\right] = \min \sum_{i=1}^{m} w_i(x-a_i)^2 + \min \sum_{i=1}^{m} w_i(y-b_i)^2$$
$$\Rightarrow \frac{d}{dx} \sum_{i=1}^{m} w_i(x-a_i)^2 = 0 = \cancel{2}\sum_{i=1}^{m} w_i(x-a_i) \Rightarrow x = \sum w_i a_i / \sum w_i$$
$$\frac{d}{dy} \sum_{i=1}^{m} w_i(y-b_i)^2 = 0 = \cancel{2}\sum_{i=1}^{m} w_i(y-b_i) \Rightarrow y = \sum w_i b_i / \sum w_i.$$

2.2.5 Thm (explicit solution formula for $1/\mathbb{R}^2/\cdot/l_2^2/\Sigma w_i$): Let $V=\{(a_i,b_i)\}_{i=1}^{m}$
$$\operatorname*{argmin}_{(x,y)\in\mathbb{R}^2} 1/\mathbb{R}^2/\cdot/l_2^2/\Sigma w_i = \left(\sum_{i=1}^{m} w_i a_i, \sum_{i=1}^{m} w_i b_i\right) \Big/ \sum_{i=1}^{m} w_i. \quad \square$$

2.2.5 Ex. (l_2): $V=\{(a_i,b_i)\}_{i=1}^{m} = \{v_i\}_{i=1}^{m}$, $1/\mathbb{R}^2/\cdot/l_2/\Sigma$
$$\min_{p=(x,y)\in\mathbb{R}^2} f(p) = \min_{p=(x,y)\in\mathbb{R}^2} \sum_{i=1}^{m} \sqrt{(x-a_i)^2+(y-b_i)^2} = \min_{p\in\mathbb{R}^2} \sum_{i=1}^{m} \|p-v_i\|_2$$

- f is convex (as a sum of convex functions)
- f is differentiable at $p \notin V$ and

$$\nabla f(p) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)(p) = \left(\sum \frac{1}{\cancel{2}}\frac{\cancel{2}(x-a_i)}{\|p-v_i\|_2}, \sum \frac{y-b_i}{\|p-v_i\|_2}\right) = \sum_{i=1}^{m} \frac{p-v_i}{\|p-v_i\|_2}$$

• $\nabla f(p) = \sum_{i=1}^{m} \frac{p-v_i}{\|p-v_i\|_2} = 0 \to \sum \frac{p}{\|p-v_i\|_2} = p \underbrace{\sum_{i=1}^{m} \frac{1}{\|p-v_i\|_2}}_{=:\lambda(p)} = \sum \frac{v_i}{\|p-v_i\|_2}$

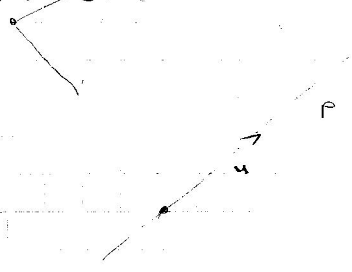
$\Rightarrow p = \frac{1}{\lambda(p)} \sum_{i=1}^{m} \frac{v_i}{\|p-v_i\|_2} =: T(p)$

• $p_1, p_2 := T(p_1), p_3 := T(p_2), \dots \to p^* = \operatorname{argmin}_{p \in \mathbb{R}^2} \sum_{i=1}^{m} \|p-v_i\|_2$

Thm 22.6 (Weiszfeld [1937]): Let $V = \{v_i\}_{i=1}^{m} \subseteq \mathbb{R}^2$ be a set of points s.t. $\dim V > 1$ (i.e. the points v_i do not lie on a line). Then there exists a unique point

$$p^* = \operatorname{argmin}_{p \in \mathbb{R}^2} \sum_{i=1}^{m} \|p-v_i\|_2 =: \operatorname{med}_{\|\cdot\|_2} \{v_i\}_{i=1}^{m}$$

and p^* is characterized as follows:



a) $p \notin V \iff \sum_{i=1}^{m} \frac{p-v_i}{\|p-v_i\|_2} = 0$

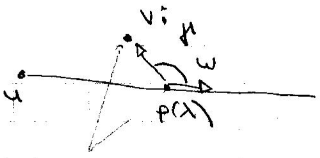
b) $p = v_k \iff \left\| \sum_{i=1, i \neq k}^{m} \frac{p-v_i}{\|p-v_i\|_2} \right\|_2 \leq 1.$

c) $p_1 \notin V,\ p_{i+1} = T(p_i),\ i = 1,2,\dots \Rightarrow (p_i) \to p^*$ if $p_i \notin V\ \forall i$.

Lem. 22.7: Let $p(\lambda) = u + \lambda w$, $\lambda \in \mathbb{R}$, be a line in $\mathbb{R}^2$, $p(\mathbb{R}) \not\supseteq V$. Then

$$g : \mathbb{R} \to \mathbb{R},\ \lambda \mapsto w^T \sum_{i=1}^{m} \frac{v_i - p(\lambda)}{\|v_i - p(\lambda)\|_2}$$

is strictly monotonously decreasing.



Proof: $w^T \frac{v_i - p(\lambda)}{\|v_i - p(\lambda)\|_2} = \|w\|_2 \cos \sphericalangle (w, v_i - p(\lambda))$ decreases and strictly for at least one v_i not on the line □

Lem. 22.8: There is at most one point satisfying 22.6 a).

Proof: Suppose $p_1, p_2 \in \mathbb{R}^2$, $p_1 \neq p_2$ do and consider $p(\lambda) = p_1 + \lambda(p_2 - p_1)$

$$\sum_{i=1}^{m} \frac{p_1 - v_i}{\|p_1 - v_i\|_2} = \sum_{i=1}^{m} \frac{p_2 - v_i}{\|p_2 - v_i\|_2} = 0$$

$p_1 + 0(p_2 - p_1)$ $\quad = p_1 + 1(p_2 - p_1)$

$$\Rightarrow (p_2 - p_1)^T \sum_{i=1}^{m} \frac{p_1 - v_i}{\|p_1 - v_i\|_2} = (p_2 - p_1)^T \sum_{i=1}^{m} \frac{p_2 - v_i}{\|p_2 - v_i\|_2} \quad ↯. \ □$$

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Lem. 22.9: If p satisfies 22.6 a), there is no v_i satisfying 22.6 b).

Proof: Suppose v_1 does and consider $p(\lambda) = v_1 + \lambda \left(\frac{p - v_1}{\|p - v_1\|_2}\right) \not\supseteq \{v_2, \dots, v_m\}$

$\Rightarrow \lambda \mapsto \frac{(p-v_1)^T}{\|p-v_1\|_2} \sum_{i=2}^{m} \frac{v_i - p(\lambda)}{\|v_i - p(\lambda)\|_2}$ is strictly mon. decreasing

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$$\to \frac{(p-v_1)^T}{\|p-v_1\|_2} \sum_{i=2}^{m} \frac{v_i - v_1}{\|v_i - v_1\|_2} > \underbrace{\frac{(p-v_1)^T}{\|p-v_1\|_2} \underbrace{\left[\sum_{i=2}^{m} \frac{v_i - p}{\|v_i - p\|_2} + \frac{v_1 - p}{\|v_1 - p\|_2} \right]}_{=0}}_{\neq 1} - \underbrace{\frac{(p-v_1)^T}{\|p - v_1\|_2} \frac{v_1 - p}{\|v_1 - p\|_2}}_{=+1}$$

↯ □

Lem. 2.2.10: At most one v_i satisfies 2.2.6 b).

Proof: Suppose v_1, v_2 do and consider $p(\lambda) = v_1 + \lambda \frac{v_2 - v_1}{\|v_2 - v_1\|_2}$. Then $p(\lambda) \not\ni \{v_3, \dots, v_m\}$ and $\frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \sum_{i=3}^m \frac{v_i - v_1}{\|v_i - v_1\|_2} > \frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \sum_{i=3}^m \frac{v_i - v_2}{\|v_i - v_2\|_2}$

$$\Rightarrow 1 \geq \left\| \sum_{i=2}^m \frac{v_i - v_1}{\|v_i - v_1\|_2} \right\|_2 \geq \frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \sum_{i=2}^m \frac{v_i - v_1}{\|v_i - v_1\|_2} = \underbrace{\frac{\|v_2 - v_1\|_2^2}{\|v_2 - v_1\|_2^2}}_{=+1} + \underbrace{\frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \sum_{i=3}^m \frac{v_i - v_1}{\|v_i - v_1\|_2}}_{\leq 0}$$

$$\Rightarrow \frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \sum_{i=3}^m \frac{v_i - v_2}{\|v_i - v_2\|_2} < 0$$

$$\Rightarrow \underbrace{\underbrace{\frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \left(\frac{v_1 - v_2}{\|v_1 - v_2\|_2}\right.}_{=-1} + \underbrace{\left.\sum_{i=3}^m \frac{v_i - v_2}{\|v_i - v_2\|_2} \right)}}_{<0} \Rightarrow \left\| \sum_{\substack{i=1 \\ i \neq 2}}^m \frac{v_i - v_2}{\|v_i - v_2\|_2} \right\|_2 > 1 \quad ⚡ \quad \square$$

Lem. 2.2.11: $\sum_{i=1}^m \|p_{j+1} - v_i\|_2 \leq \sum_{i=1}^m \|p_j - v_i\|_2$, $j = 1, 2, \dots$ and "=" $\Leftrightarrow p_{j+1} = p_j$

Proof: Let $w_i := \frac{1}{\|p_j - v_i\|_2}$, $i = 1, \dots, m$

Thm. 2.2.5 $\Rightarrow$ $\underset{p \in \mathbb{R}^2}{\operatorname{argmin}} \sum_{i=1}^m w_i \|p - v_i\|_2^2 = \frac{1}{\underbrace{\sum_{i=1}^m w_i}_{=\lambda(p_j)}} \sum_{i=1}^m w_i v_i = \frac{1}{\lambda(p_j)} \sum_{i=1}^m \frac{1}{\|p_j - v_i\|_2} v_i$

$$= T(p_j) = p_{j+1}$$

$$\Rightarrow \cancel{\sum_{i=1}^m \|p_j - v_i\|_2} = \sum_{i=1}^m \underbrace{w_i}_{=\frac{1}{\|p_j - v_i\|_2}} \|p_j - v_i\|_2^2 \geq \sum_{i=1}^m \frac{1}{\|p_j - v_i\|_2} \underbrace{\|p_{j+1} - v_i\|_2^2}$$

$$= \left[\|p_j - v_i\|_2 + (\|p_{j+1} - v_i\|_2 - \|p_j - v_i\|_2) \right]^2$$

$$= \|p_j - v_i\|_2^2 + 2 \|p_j - v_i\|_2 (\|p_{j+1} - v_i\|_2 - \|p_j - v_i\|_2) + (\|p_{j+1} - v_i\|_2 - \|p_j - v_i\|_2)^2$$

$$= \cancel{\sum \|p_j - v_i\|_2} + 2 \sum \|p_{j+1} - v_i\|_2 - 2 \sum \|p_j - v_i\|_2 + \sum \frac{(\|p_{j+1} - v_i\|_2 - \|p_j - v_i\|_2)^2}{\|p_j - v_i\|_2}$$

$$\Rightarrow 2 \sum \|p_j - v_i\|_2 \geq 2 \sum \|p_{j+1} - v_i\|_2 + \underbrace{\sum (\|p_{j+1} - v_i\|_2 - \|p_j - v_i\|_2)^2 / \|p_j - v_i\|_2}_{\geq 0} \quad \square$$

Lem. 2.2.12: (p_i) is bounded and hence has accumulation points

Proof: $p_i \in \operatorname{conv} V$, $i = 2, 3, \dots$ (as centers of gravity). $\square$

Lem. 2.2.13: $T(p) = p$ for any accumulation point $p \notin V$ of (p_i).

Proof: Suppose $T(p) = p'$, then p' is also an accumulation point of (p_i) as T is continuous (images of points near p are near p'), and $\sum \|p - v_i\| > \sum \|p' - v_i\|$, while $\sum \|p_j - v_i\| \to \sum \|p - v_i\| = \sum \|p' - v_i\|$ ⚡ $\square$

Lem. 2.2.14: Any accumulation point $p \notin V$ of (p_j) satisfies 2.2.6 a).

Proof: Let $p = (x, y)$. Then $p = T(p)$

$$\Rightarrow x = \frac{\sum \frac{a_i}{\|p - v_i\|_2}}{\sum \frac{1}{\|p - v_i\|_2}} \iff \frac{\sum \frac{x - a_i}{\|p - v_i\|_2}}{\sum \frac{1}{\|p - v_i\|_2}} = 0 \Rightarrow \sum \frac{p - v_i}{\|p - v_i\|_2} = 0. \square$$

Cor. 2.2.15: (p_i) admits at most one accumulation point $p \notin V$.

Proof: Lem. 2.2.8 states uniqueness. $\square$

Lem. 2.2.15: If v_i is an accumulation point of (p_i), it is the only condensation point.

Proof: W.l.o.g. let $v_i = v_1$. Other accumulation points can only be $v_2, \dots, v_m$ and the unique point p satisfying 2.2.6 a). (if it exists) $\Rightarrow \exists U_\varepsilon(v_1) \not\ni \{v_2, \dots, v_m, p\}$.

Consider $j_1, j_2, \dots$ s.t. $p_{j_k} \in U_\varepsilon(v_1)$, $p_{j_k+1} \notin U_\varepsilon(v_1)$, $k = 1, 2, \dots$

$$\Rightarrow \|p_{j_k} - v_1\|_2 \to 0, \quad \|p_{j_k} - v_i\|_2 \to \|v_1 - v_i\|_2, \quad i = 2, \dots, m \qquad (1)$$

$$\Rightarrow \frac{\|p_{j_k+1} - v_1\|_2}{\|p_{j_k} - v_1\|_2} > \frac{\varepsilon}{\|p_{j_k} - v_1\|_2} \to \infty$$

W.l.o.g. let $v_1 = 0$. Then

$$\frac{\|p_{j_k+1} - v_1\|_2}{\|p_{j_k} - v_1\|_2} = \frac{\|T(p_{j_k})\|_2}{\|p_{j_k}\|_2} = \frac{\left\| \sum_{i=2}^m \frac{v_i}{\|p_{j_k} - v_i\|_2} \right\|_2}{\underbrace{\sum_{i=1}^m \frac{1}{\|p_{j_k} - v_i\|_2}}_{= \frac{1}{\|p_{j_k}\|_2} + \sum_{i=2}^m \frac{1}{\|p_{j_k} - v_i\|_2}}} \cdot \frac{1}{\|p_{j_k}\|_2}$$

$$= \frac{\left\| \sum_{i=2}^m \frac{v_i}{\|p_{j_k} - v_i\|_2} \right\|_2 \to \|v_i\|_2}{1 + \sum_{i=2}^m \frac{\|p_{j_k}\|_2 \to 0}{\|p_{j_k} - v_i\|_2 \to \|v_1 - v_i\|_2 = \|v_i\|_2}}$$

$$\to \left\| \sum_{i=2}^m \frac{v_i}{\|v_i\|_2} \right\|_2 =: \nu < \infty \ \lightning. \ \square \qquad (2)$$

Lem. 2.2.16: If v_i is an accumulation point of (p_i), it satisfies 2.2.6 b).

W.l.o.g. let $v_i = v_1 = 0$. By Lem. 2.2.15, v_1 is the only accumulation point of (p_i)

$$\Rightarrow p_j \to v_1 \Rightarrow \|p_j - v_1\|_2 \to 0$$

$$(1) \Rightarrow \|p_j - v_i\|_2 \to \|v_1 - v_2\|_2 \qquad p_j \to v_1$$

$$(2) \Rightarrow \frac{\|p_{j+1} - v_1\|_2}{\|p_j - v_1\|_2} \to \left\| \sum_{i=2}^m \frac{v_i}{\|v_i\|_2} \right\|_2 = \nu < 1 \quad (2.2.6\,b)) \ \square$$

$$= \left\| \sum_{i=2}^m \frac{v_1 - v_i}{\|v_1 - v_i\|_2} \right\|_2$$

Lem. 22.17: Suppose $p_j \notin V$, $j=1,2,\dots$ Then (p_j) converges to a point p^* satisfying 22.6 a) or b). Moreover, p^* does not depend on p_1.

Proof: (p_j) has an accumulation point p^* by Lem. 22.12. If $p^* \in V$, it is unique by Lem. 22.16 and satisfies 22.6 b) by Lem. 22.17. If $p^* \notin V$, it is unique by Lem. 22.15 and satisfies 22.6 a) by Lem. 22.14.

Let $p_1' \neq p_1$ and suppose $\lim p_j' = p' \neq p^*$. We have one of the following relations:

a) $\sum_{i=1}^{m} \frac{p^*-v_i}{\|p^*-v_i\|_2} = 0$, $\sum_{i=1}^{m} \frac{p'-v_i}{\|p'-v_i\|_2} = 0 \overset{\text{Lem. 22.8}}{\Rightarrow} ↯$

b) " , $\left\| \sum_{i=1}^{m} \frac{p'-v_i}{\|p'-v_i\|_2} \right\|_2 \le 1 \overset{\text{Lem. 22.9}}{\Rightarrow} ↯$

c) $\left\| \sum_{i=1}^{m} \frac{p^*-v_i}{\|p^*-v_i\|_2} \right\|_2 \le 1$, $\sum_{i=1}^{m} \frac{p'-v_i}{\|p'-v_i\|_2} = 0 \overset{\text{Lem. 22.9}}{\Rightarrow} ↯$

d) " , $\left\| \sum_{i=1}^{m} \frac{p'-v_i}{\|p'-v_i\|_2} \right\|_2 \le 1 \overset{\text{Lem. 22.10}}{\Rightarrow} ↯$ □

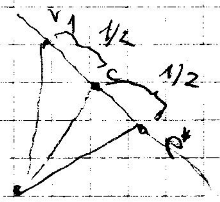
Prop. 22.18: $\lim p_j = p^* = \arg\min \sum_{i=1}^{m} \|p-v_i\|_2$ if $p_j \notin V$, $j=1,2,\dots$

Proof: a) For $x \neq p^*$, $x \notin V$, let $x_1 = x$, $x_2 = T(x_1)$, ...

$$\sum \|\underbrace{x}_{=x_1} - v_i\|_2 \overset{\text{Lem. 22.11}}{>} \sum \|x_2 - v_i\|_2 > \dots = \lim \sum \|x_j - v_i\|_2 = \sum \|p - v_i\|_2 .$$

b) For $x = v_i \neq p^*$, w.l.o.g. $x = v_1 \neq p^*$, suppose $\sum \|\underbrace{x}_{=v_1} - v_i\|_2 \le \sum \|p^* - v_i\|_2$ (3

Let c be the median of v_1 and p^* and consider the



Δ v_1, p^*, v_i, $i=2,\dots,m$. In any Δ,

$\|v_i - c\|_2 \le (\|v_i - v_1\|_2 + \|v_i - p^*\|_2)/2$ and "=" $\Leftrightarrow v_i \in \text{lin}\{v_1, p^*\}\ \forall i$

$$\Leftrightarrow \sum \|c - v_i\|_2 \le \frac{\sum \|v_1 - v_i\|_2 + \sum \|p^* - v_i\|_2}{2} \le \sum \|p^* - v_i\|_2$$

If $c \notin V \overset{a)}{\Rightarrow} ↯$. If $c \in V$ consider $c' = \text{med}\{c, p^*\}$ and so on until $c' \notin V$ □

Rem. 22.19: a) $p_j \in V$ can be repaired (e.g. by choosing p_1 appropriately).

b) There are solution formulas for $m=3,4$. $m=5$ cannot be solved by a formula involving only elementary algebraic operations ($+/-$, $\times/\div$, $\sqrt{\ }$).