3.3. Local Search

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3.3.1 Def. (Neighborhood, Swap for the metric p-median problem): given $V = J \subseteq \mathbb{R}^k$, consider the metric p-median problem $p/I/\cdot/\|\cdot\|/\Sigma$:

a) $\mathcal{S} := \{S \subseteq I : |S| = p\}$ set of p-medians (feasible solutions)

b) $c(S) := \sum_{j \in J} \|s_j, j\| = \sum_{j \in J} \min_{i \in S} \|i, j\|$ cost of a solution

c) $\Gamma(S) := \{S' \in \mathcal{S} : |S' \Delta S| = 2\}$ neighborhood of a solution

d) $i \to i' : \{S \in \mathcal{S} : i \in S, i' \notin S\} \to \mathcal{S}$, $S \mapsto S \setminus \{i\} \cup \{i'\}$ swap

e) $S \in \mathcal{S}$ local optimum $:\Leftrightarrow \{S' \in \Gamma(S) : c(S') < c(S)\} = \emptyset$.

3.3.3 Alg. (Local Search for the metric p-median problem):

Input: $p/I/\cdot/\|\cdot\|/\Sigma$, $V = J$, $S \in \mathcal{S}$

Output: $\bar{S} \in \mathcal{S}$ local optimum

1. while $\exists S' \in \Gamma(S) : c(S') < c(S)$

   $S \leftarrow S'$

2. output S.

3.3.2 Rem. $p/I/\cdot/\|\cdot\|/\Sigma \Leftrightarrow$ (MUFL), $\sum_{i \in I} y_i = p$, $f \equiv$ const.

3.3.4 Rem.: Let $\bar{S}$ be a local, S^* a global optimum. We want to know:

a) How large can $c(\bar{S}) / c(S^*)$ be?

b) How fast does Alg. 3.3.3 converge?

Arya, Garg, Khandekar, Munagala & Pandit [2001] showed

a) $c(\bar{S}) \leq 3\, c(S^*)$

b) $\forall S \in \mathcal{S} : \exists S' \in \Gamma(S) : \underbrace{c(S) - c(S')}_{\text{decrease per swap}} \geq \dfrac{c(S) - 3\,c(S^*)}{n^2}$

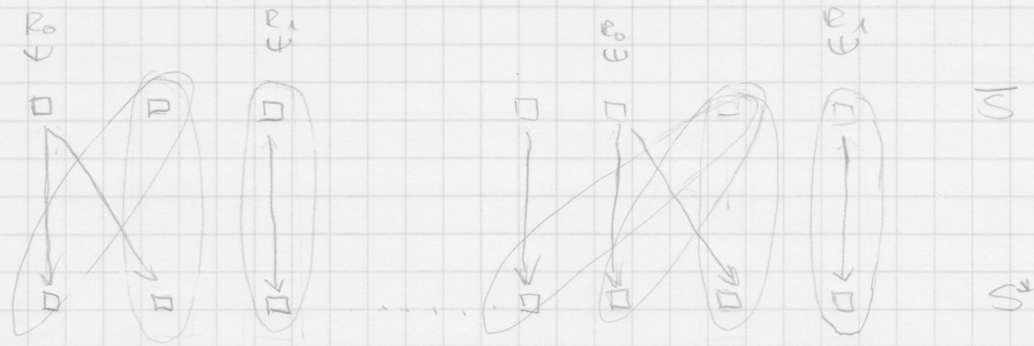
and that this yields an $3 + O(1)$-approximate polynomial time local search algorithm.

3.3.5 Thm (5-approximation ratio for local search): Alg. 3.3.3 yields a local optimum $\bar{S}$ such that $c(\bar{S}) \leq 5\, c(S^*)$.

Proof: Let $\bar{S}$ be the output of Alg. 3.3.3, S^* an optimal solution. Define a map

$$\eta : S^* \to \bar{S}, \quad i^* \mapsto \operatorname{argmin}_{i \in \bar{S}} \|i, i^*\|,$$

that maps each facility in S^* to the closest facility in $\bar{S}$, b.l.o.a. Let



$R_k := \{ i \in \bar S : \eta(i^*) = i$ for exactly k facilities $i^* \in S^* \}$, $k = 0, 1$.

Construct a set P of k swaps, one for each $i^* \in S^*$, as follows:

a) $i \in R_1 \Rightarrow i \to \eta^{-1}(i) \in P$

b) $|R_0| + \underbrace{|\bar S \setminus (R_0 \cup R_1)|}_{\le \frac{1}{2} |S^* \setminus \eta^{-1}(R_1)|} = |S^* \setminus \eta^{-1}(R_1)| \ge \frac{1}{2} |S^* \setminus \eta^{-1}(R_1)|$

$\Rightarrow 2|R_0| \ge |S^* \setminus \eta^{-1}(R_1)|$

$i \in R_1 \Rightarrow i \to i^* \in P$ for at most 2 arbitrarily chosen $i \in S^* \setminus \eta^{-1}(R_1)$

Rem.: a) $i \in R_1$ is close to $\eta^{-1}(i)$, all other $i^* \in S^*$ are far away

$\Rightarrow i \to \eta^{-1}(i)$ can be handled by assigning all of i's clients to $\eta^{-1}(i)$

b) $i \in \bar S \setminus (R_0 \cup R_1)$ is close to several facilities $\eta^{-1}(i)$.

$\Rightarrow i \to i^* \in \eta^{-1}(i)$ and assigning all of i's clients to i^* can

cost much, so we consider only swaps $i \to i^* \in P$.

Claim 1: For $i \in \bar S$, $i^*, i^{*\prime} \in S^*$, and $i \to \overset{R_0 \cap R_1}{i^*} \in P$ it holds $\eta(i^{*\prime}) \neq i$.

Let

$\bar\varphi : J \to \bar S$, $j \mapsto \arg\min \|\bar S, j\|$

$\varphi^* : J \to S^*$, $j \mapsto \arg\min \|S^*, j\|$

functions mapping j to the closest facility in $\bar S$ and S^*, respectively,

$\bar N(i) = \bar\varphi^{-1}(i)$, $i \in \bar S$

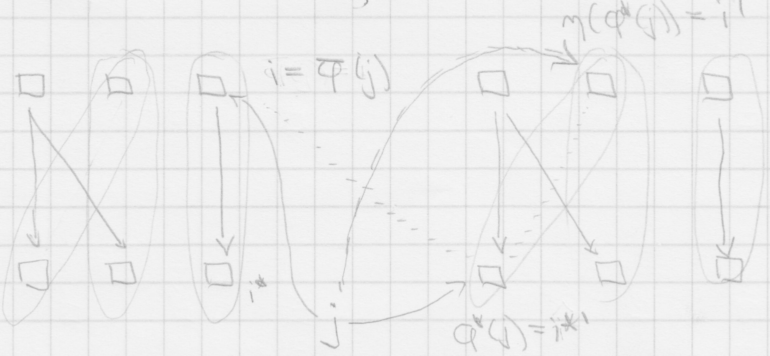
$N^*(i) = \varphi^{*-1}(i)$, $i \in S^*$

be the sets of clients assigned to $i \in \bar S$ and $i \in S^*$, respectively.

Claim 2: For each swap $i \to i^* \in P$ it holds

$$0 \overset{(1)}{\le} c(\bar S \setminus \{i\} \cup \{i^*\}) - c(\bar S) \overset{(2)}{\le} \sum_{j \in N^*(i^*)} \left(\|S^*, j\| - \|\bar S, j\| \right) + \sum_{j \in N(i)} 2 \|S^*, j\|.$$

Subst.: (1) is because $\bar S$ is a local optimum



Consider the following assignment $\varphi' : J \to \overline{S} \setminus \{i\} \cup \{i^*\}$

$$\varphi'(j) = \begin{cases} i^*, & j \in N^*(i^*) \\ i' = \eta(\varphi^*(j)), & j \in \overline{N}(i) \setminus N^*(i^*) \\ \overline{\varphi}(j), & j \notin N^*(i^*), \overline{N}(i) \end{cases}$$

Because of claim 1, $i \neq i'$, so φ' is well defined. Further

$$c(\overline{S} \setminus \{i\} \cup \{i^*\}) - c(\overline{S}) = \sum_{j \in J} \left(\| \overline{S} \setminus \{i\} \cup \{i^*\}, j \| - \| \overline{S}, j \| \right)$$

$$= \sum_{j \in N^*(i^*)} \left(\underbrace{\| i^*, j \|}_{= \| S^*, j \|} - \| \overline{S}, j \| \right) + \sum_{j \in \overline{N}(i) \setminus N^*(i^*)} \underbrace{\left(\| i', j \| - \| i, j \| \right)}$$

$$\leq \| i^{*\prime}, j \| + \underbrace{\underbrace{\| i^{*\prime}, i' \|}_{\leq \| i^{*\prime}, i \|} - \| i, j \|}_{\leq \| i^{*\prime}, j \|}$$

$$\leq 2 \| i^{*\prime}, j \|$$

$$= 2 \| S^*, j \|$$

$$= \sum_{j \in N^*(i^*)} \left(\| S^*, j \| - \| \overline{S}, j \| \right) + \sum_{j \in \overline{N}(i) \setminus N^*(i^*)} 2 \| S^*, j \|$$

$$\leq \quad '' \quad + \sum_{j \in \overline{N}(i)} 2 \| S^*, j \|$$

Summing over all $i \to i^* \in P$ gives

$$0 \leq \sum_{i^* \in S^*} \sum_{j \in N^*(i^*)} \left(\| S^*, j \| - \| \overline{S}, j \| \right) + 2 \sum_{i \in \overline{S}} \sum_{j \in \overline{N}(i)} 2 \| S^*, j \|$$

i appears at most 2 times

$$= 5 c(S^*) - c(\overline{S})$$

$$\Rightarrow c(\overline{S}) \leq 5 c(S^*).$$