1 Checking a gas nomination for feasibility

- problem definition
  - input: gas network graph $G = (V, arcs)$, balanced nomination $\boldsymbol{\omega}$ (vector of gas in- and outflows)
  - task: decide whether gas flow can technically be realized
  - most important variables: flow q_a along an arc
  - flow conservation constraints: $A\boldsymbol{q} = \boldsymbol{\omega}$
  - difference from "linear" flow problems: no capacities on arcs, but q_a at arc (u,v) induced by pressures at nodes u and v
    pressure at node u: p_u
  - important technical constraints: bounds on the pressures
  - relationship between q_a and p_u and p_v depends on type of arc (pipe, compressor, control valve, valve)
  - pipe: most accurately modelled by a PDE, good algebraic approximations known, e. g.,

$$p_v^2 = \left(p_u^2 - \Lambda_a \left| q_a \right| q_a \frac{\mathrm{e}^{S_a} - 1}{S_a} \right) \mathrm{e}^{-S_a} \tag{1}$$

    with constants

Λ_a	modelling physical properties of the gas, the pipe and its environment, e. g., length, diameter, roughness, soil temperature
S_a	modelling the height difference

  - compressor (increase pressure): feasible set of (q_a, p_u, p_v) is nonconvex; can be approximated by intersection of 4 quadratic constraints
  - control valve (decrease pressure): nice, feasible set of (q_a, p_u, p_v) is 2-dimensional interval
  - valve: two states open or closed $\rightsquigarrow$ binary variable z_a

$$\begin{aligned} z_a = 0 \Longrightarrow \quad & q_a = 0; & & p_u,\ p_v \text{ arbitrary}, \\ z_a = 1 \Longrightarrow \quad & q_a \text{ arbitrary}; & & p_u = p_v. \end{aligned}$$

  - valves are used to route the gas in the network, resulting in complex overall behavior
- problem is a nonconvex MINLP which we want to solve globally
- global optimization basically only possible for convex (or even linear) problems
- ways out:
  - use convex underestimators and spatial branching

- linearize nonlinear functions
- exploit special problem structure

- for checking nomination feasibility, we use two competing approaches
  1. construct a MIP approximation to be solved by a standard MIP solver
  2. use a MINLP approximation with special structure to be solved by a custom-tailored solver (based on SCIP)

2 Constructing MIP approximations of MINLPs

- step 1: convert function to a piecewise linear one
- step 2: use a MIP model for piecewise linear functions to incorporate it in a MIP
- for simplicity, consider only univariate continuous piecewise linear functions
- Remark: It is often possible to rewrite multivariate functions as combinations of univariate functions.
- example: reformulating (1):

$$\begin{aligned} p_v^2 &= \left(p_u^2 - \Lambda_a \left| q_a \right| q_a \frac{\mathrm{e}^{S_a} - 1}{S_a} \right) \mathrm{e}^{-S_a} \\ \mathrm{e}^{-S_a} p_u^2 - p_v^2 &= \Lambda_a \left| q_a \right| q_a \frac{\mathrm{e}^{S_a} - 1}{S_a} \mathrm{e}^{-S_a} \end{aligned}$$

  introducing a new variable Δ_{uv} for the pressure drop (lhs) we may write

$$\begin{aligned} \Delta_{uv} &= \Lambda_a \left| q_a \right| q_a \frac{\mathrm{e}^{S_a} - 1}{S_a} \mathrm{e}^{-S_a} \\ p_v^2 &= \mathrm{e}^{-S_a} p_u^2 - \Delta_{uv} \end{aligned}$$

- notation:
  - $f\colon \mathbb{R} \to \mathbb{R}$: continuous piecewise linear function
  - $x_0, \ldots, x_n$: endpoints of the n intervals in which f is linear
  - $f_i = f(x_i)$, $0 \leq i \leq n$: function value at x_i

2.1 The convex combination method

- idea: in each interval $[x_i, x_{i+1}]$, the exact value of f is given by a convex combination of f_i and f_{i+1}
- introduce variables $\lambda_i \geq 0$, $0 \leq i \leq n$, to describe convex combinations

- basic model:

$$\begin{aligned} \lambda_i &\geq 0 && 0 \leq i \leq n, \\ \sum_{i=0}^{n} \lambda_i &= 1, \\ x &= \sum_{i=0}^{n} \lambda_i x_i, \\ f &= \sum_{i=0}^{n} \lambda_i f_i. \end{aligned}$$

- missing property:

At most two of the λ_i may be positive. If two λ_i are positive, they need to be adjacent. (SOS2)

- enforce (SOS2) via additional binary variables z_i, $1 \leq i \leq n$, indicating which interval is used (i. e., $z_i = 1$ iff $x \in [x_{i-1}, x_i]$):

$$\begin{aligned} z_i &\in \{0,1\} && 1 \leq i \leq n, \\ \sum_{i=1}^{n} z_i &= 1, \\ \lambda_0 &\leq z_1, \\ \lambda_i &\leq z_i + z_{i+1} && 1 \leq i < n, \\ \lambda_n &\leq z_n. \end{aligned}$$

2.2 The SOS method

- idea: enforce (SOS2) via branching instead of additional binary variables
- model contains no binary variables, but a constraint
  "The set $\{\lambda_0, \ldots, \lambda_n\}$ is a SOS2 set."
- special branching rule for this type of constraint:
  1. Compute
  $$w = \frac{\sum_{i=0}^{n} i\lambda_i}{\sum_{i=0}^{n} \lambda_i}.$$
  2. There is a unique pair $(k, k+1)$ with $k \leq w \leq k+1$.
  3. Branch via
  $$\sum_{i=0}^{k} \lambda_i = 0 \qquad \text{and} \qquad \sum_{i=k}^{n} \lambda_i = 0$$
- of course, a scoring mechanism to balance with the "usual" branching is needed

2.3 The log method

- Let $I = \{1, \dots, n\}$ be the index set of the intervals and $J = \{0, \dots, n\}$ be the index set for the λ-variables.
- Define the set $I(j)$ by

$$I(j) = \begin{cases} \{1\} & j = 0, \\ \{j, j+1\} & 1 \le j < n, \\ \{n\} & j = n. \end{cases}$$

- Assume $n = 2^k$ for some $k \ge 2$ for simplicity.
- We need to decide which of the n intervals to chose. Can we do this using only k binary variables?

Definition 1 A bijective function $B\colon \{1, \dots, 2^k\} \to \{0,1\}^k$ is called a *Gray code*, if $B(j)$ and $B(j+1)$ differ in exactly one component.

Theorem 1 *Let $B\colon I \to \{0,1\}^k$ be a Gray code. The constraints*

$$\sum_{j \in J\colon\, B(i)_l = 1\, \forall i \in I(j)} \lambda_j \le x_l \qquad 1 \le l \le k, \tag{2}$$

$$\sum_{j \in J\colon\, B(i)_l = 0\, \forall i \in I(j)} \lambda_j \le 1 - x_l \qquad 1 \le l \le k, \tag{3}$$

$$x_l \in \{0,1\} \qquad 1 \le l \le k, \tag{4}$$

are a MIP model for (SOS2).

Proof
- Intuition: λ_j has to be zero if $x = (x_1, \dots, x_k)$ is different from $B(i)$ for any interval $i \in I(j)$
- Let (λ, x) be an integer solution of the model.
- Need to show for $j \in J$: If $x \ne B(i)$ for $i \in I(j)$, then $\lambda_j = 0$.
- Case $j \in \{0, n\}$:
  - We have $I(0) = \{1\}, I(n) = \{n\}$. Thus λ_j appears in the LHS of (2) or (3) for any l.
  - $B(j) \ne x$ implies there is a l with $B(j)_l \ne x_l$. Thus there is a constraint (2) or (3) with RHS 0, where λ_j appears on the LHS, thus forcing it to 0.
- Case $j \in J \setminus \{0, n\}$:
  - We have $I(j) = \{j, j+1\}$, so λ_j appears on the LHS iff $B(j)_l = B(j+1)_l$.
  - Since B is a Gray code, $x \notin \{B(j), B(j+1)\}$ implies there is a l with $B(j)_l = B(j+1)_l \ne x_l$. Thus there is again a constraint with RHS 0 and λ_j on the LHS.

3 Solving a specially-structured MINLP approximation

- Consider purely passive gas network, consisting of pipes only (in particular, there are no valves / discrete decisions).
- Assuming that the height difference of all pipes is negligible, the pressure drop model further simplifies to

$$p_u^2 - p_v^2 = \Lambda_a \, |q_a| \, q_a.$$

- Replacing p_u^2 by new variables $\pi_u := p_u^2$, the feasibility checking problem of a nomination $\boldsymbol{\omega}$ is then of the form:

$$\begin{aligned} \sum_{a \in d^+(v)} q_a - \sum_{a \in d^-(v)} q_a &= d_u \qquad && \forall u \in V, \\ \pi_u - \pi_v &= \Lambda_a \, |q_a| \, q_a \qquad && \forall a \in A. \end{aligned} \tag{5}$$

Theorem 2 *The solution set of* (5) *has the following properties:*

- *The flows* $\boldsymbol{q} = (q_a)_{a \in A}$ *are unique.*
- *The set of feasible pressure squares* $\boldsymbol{\pi} = (\pi_u)_{u \in V}$ *is a line: If* $\boldsymbol{\pi}_0$ *is feasible, so is* $\boldsymbol{\pi}_0 + \lambda \mathbb{1}$*, where* $\mathbb{1}$ *denotes the* $|V|$*-dimensional vector of 1s.*

- NB: The solution set of (5) is thus convex.
- To take into account the pressure bounds $\underline{\pi}_u$, $\overline{\pi}_u$ at each node u, we introduce additional slack variables $s_u^\pi \in \mathbb{R}_{\geq 0}$ and the constraints

$$\begin{aligned} \pi_u + s_u^\pi &\geq \underline{\pi}_u \quad && \forall u \in V, \\ \pi_u - s_u^\pi &\leq \overline{\pi}_u \quad && \forall u \in V. \end{aligned} \tag{6}$$

- Minimizing

$$\sum_{u \in V} s_u^\pi$$

  over the flow conservation constraints, (5), and (6) is then a convex NLP, which can be solved to global optimality. Its objective value is 0 if and only if the nomination $\boldsymbol{d}$ can be realized by the passive gas network without violating the pressure bounds.

- Motivates the following approach:
  - Reformulate constraints for compressors and control valves in way compatible to (a generalization of) Theorem 2.
  - Resort to above NLP as soon as valves have been decided to check feasibility.