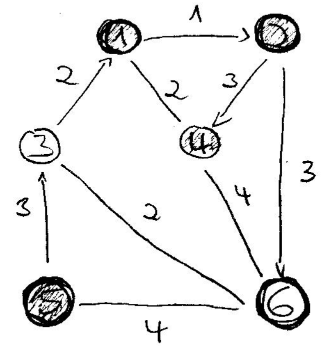
3. The Uncapacitated Facility Location Problem

3.1. The general Uncapacitated Fac. Loc. Prob.

3.1.1 Ex. (Schöbel & Schmidt [2008]) & Def. (Uncapacitated Fac. Loc. Problem):



$G = (V, E \cup A)$ network

$I \subseteq V$ customers (clients)

$J \subseteq V$ potential facilities

$d_{ij} \in \mathbb{R}_{\geq 0}$, $ij \in A$ transportation costs $i \to j$

$f_i \in \mathbb{R}_{\geq 0}$, $i \in I$ setup costs

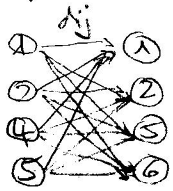
$A \subseteq I \times J$ Uncapacitated Facility Loc. Prob.:

$I = \{1, 2, 5, 6\}$

$J = \{1, 2, 4, 5\}$

$$\text{(UFL)} \quad \min_{I^* \subseteq I} f(I^*) + \sum_{j \in J} \min_{i \in I^*} d_{ij}$$

$$D = (d_{ij}) = \begin{array}{c|cccc|c|c|c} & 1 & 2 & 5 & 6 & \Sigma & p=1 & p=... \\ \hline 1 & 0 & 1 & 8 & 4 & 13 & 18 & 15 \checkmark \\ 2 & 5 & 0 & 7 & 3 & 15 & 20 & \\ 4 & 2 & 3 & 8 & 4 & 17 & 22 & \\ 5 & 6 & 3 & 0 & 4 & 13 & 18 & \checkmark \end{array}$$



associated bipartite graph $G = (I \cup J, A)$

$f = (f_i) = (5, 5, 5, 5)$

3.1.2. Prop. (IP-Formulation for the UFL, Balinski [1965]).

$y_i \in \{0, 1\}$, $i \in I$. facility setup variables

$x_{ij} \in \{0, 1\}$, $j \in J$ client assignment variables

$$\begin{array}{llll} \text{(UFL)} & \min \sum_{i \in I} f_i y_i + \sum_{ij \in A} d_{ij} x_{ij} & & \\ \text{(i)} & \sum_{i \in I} x_{ij} \geq 1 & \forall j \in J & \\ \text{(ii)} & y_i \geq x_{ij} & \forall ij \in A & \\ \text{(iii)} & y_i \in \mathbb{Z}_+ & \forall j \in J & \\ \text{(iv)} & x_{ij} \in \mathbb{Z}_+ & \forall ij \in A & \end{array}$$

a) (UFL) has an optimal 0/1-solution.

b) (UFL) $\Leftrightarrow$ (UFL)(i),(ii),(iii), $x_{ij} \geq 0$ $\forall ij \in A$, i.e., the client assignment variables are automatically integral.

Peter's Comp $\leftarrow$ **Proof.:** Ex. sheet 11. $\square$

3.1.3. Prop. (Set Covering Model for the UFL):

$J(i) := \{j \in J : ij \in A\}$ clients covered by fac. i

$\mathcal{J} := \{(i, J') : i \in I, \emptyset \neq J' \subseteq J(i)\}$

$$c_{(i,J')} := f_i + \sum_{j\in J'} d_{ij}$$

$$\text{(SCP)} \quad \min \sum_{(i,J')\in \mathcal{J}} c_{(i,J')} z_{(i,J')}$$

$$\sum_{J'\ni j} z_{(i,J')} \geq 1 \quad \forall j \in J$$

$$z_{(i,J')} \in \{0,1\} \quad \forall (i,J') \in \mathcal{J}$$

Then there is a one-to-one correspondence between optimal solutions of (SCP) and (UFL).

Proof: Ex. sheet 11. □

Cor. 3.13: a) The greedy algorithm for UFL is $H(\max |J(i)|)$-approximate

b) UFL is APX-hard.

Proof: a) Follows from Thm. 2.6.8.

SCP is APX-hard.

b) Given an SCP $\min c^T x, Mx \geq 1, x \in \{0,1\}^n$, construct an UFL in polytime

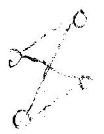
$I = \{1,\dots,n\}$, $J = \{1,\dots,m\}$, $A = \{ij : m_{ij} = 1\}$, $f_j = c_j$.

Then (UFL) $\Leftrightarrow$ (SCP) and the equivalence is approximation preserving. □

Obs. 3.14 (Relation to the p-median problem):

a) (UFL) $\Leftrightarrow$ $|I|/I/\cdot/d_{ij}/\Sigma$

b) $p/V/\cdot/d_{ij}/\Sigma$ $\Leftrightarrow$ (UFL), (V) $\sum_{i\in I} y_i \leq p$.



3.2. The Metric Uncapacitated Facility Location Problem

3.2.1 Def. (Metric UFL): A UFL s.t.

i) $I, J \subseteq \mathbb{R}^k$, $k \in \mathbb{N}$, : I, J are embedded in $\mathbb{R}^k$

ii) $A = I \times J$, all assignments are possible

iii) $d_{ij} = \|i,j\|$, distances w.r.t. some norm

is a metric UFL (MUFL).

3.2.3. Alg. (LP-Rounding):

Input: (MUFL) $I, J \subseteq \mathbb{R}^k$, $d_{ij} = \|i,j\|$ $\forall (i,j) \in I\times J$, $f_i \in \mathbb{R}_{\geq 0}$ $\forall i \in I$

Output: (x,y) feasible for (MUFL)

1. LP-Solving: $(\bar{x},\bar{y}) \leftarrow \operatorname{argmin}\,(\text{MUFL})_{LP}$, where

$$(\text{MUFL})_{LP} \quad \sum_{i\in I} f_i y_i + \sum_{ij\in A} d_{ij} x_{ij}$$

$$\sum_{i\in I} x_{ij} \ge 1 \quad \forall j\in J$$
$$y_i \ge x_{ij} \quad \forall ij\in A$$
$$y_i \ge 0 \quad \forall i\in I$$
$$x_{ij} \ge 0 \quad \forall ij\in A$$

1. Filtering:

$D_j \leftarrow \sum_{i\in I} d_{ij}\bar{x}_{ij}$ "total fractional distance"

$N_j \leftarrow \{ i\in I : \underbrace{\bar{x}_{ij}>0}_{\text{"fractionally assigned"}} \wedge \underbrace{d_{ij}\le 2D_j}_{\text{"near"}} \}$ "near frac. assigned fa

3.2.4 Claim: $\sum_{i\in N_j} \bar{x}_{ij} \ge \frac{1}{2} \quad \forall j\in J$

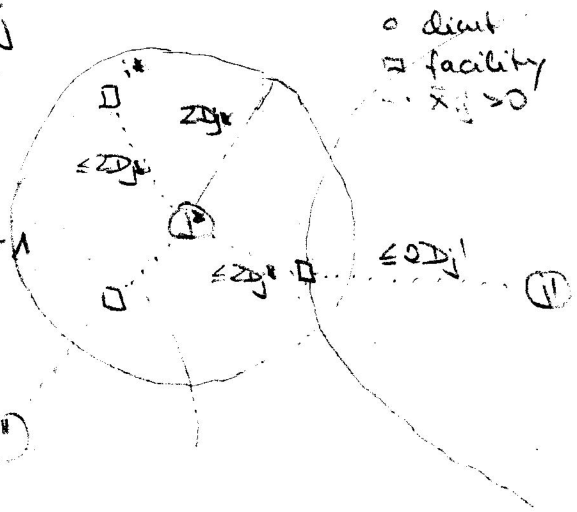
$$x'_{ij} \leftarrow \begin{cases} 0, & i\notin N_j \\ 2\bar{x}_{ij}, & \text{else} \end{cases}, \quad \forall ij\in A$$

$$y'_i \leftarrow \max_{j\in J} x'_{ij}$$

3.2.5 Obs: (x',y') is feasible for $(\text{MUFL})_{LP}$, and

i) $x'_{ij}>0 \Rightarrow d_{ij}\le 2D_j$

ii) $(x',y') \le 2(\bar{x},\bar{y})$.



2. Rounding:

2a. $J^* = \emptyset$, $(x^0,y^0)\leftarrow(x',y')$, $k\leftarrow 1$

2b. $j^* \leftarrow \arg\min_{j\in J\setminus J^*} D_j$

$i^* \leftarrow \arg\min_{i\in N_{j^*}} f_i$

$EN_{j^*} \leftarrow \{ j\in J\setminus J^* : i^*\in N_j \}$ "extended neighborhood of client j^*"

$$y_i^k \leftarrow \begin{cases} 1, & i=i^* \\ 0, & i\in N_{j^*}\setminus\{i^*\} \\ y_i^{k-1}, & \text{else} \end{cases}$$ select i^*, drop $N_{j^*}\setminus\{i^*\}$

$$x_{ij}^k \leftarrow \begin{cases} 1, & i=i^*,\ j\in EN_{j^*} \\ 0, & i\in N_{j^*}\setminus\{i^*\},\ j\in J \\ 0, & i\notin N_{j^*},\ j\in EN_{j^*} \\ x_{ij}^{k-1}, & \text{else} \end{cases}$$ assign all unass. clients in EN_{j^*} to i^*

$J^* \leftarrow J^* \cup EN_{j^*}$

goto 2b.

2c. $(x'', y'') \leftarrow (x^k, y^k)$, output (x'', y'').

32.6 Claim:

i) $j' \in EN_{j^*} \Rightarrow d_{i^* j'} \leq 2 \cdot 2D_{j^*} + 2D_{j'} \leq 6D_{j'}$ $\quad D_{j^*} \leq D_{j'}$

ii) $f_{i^*} \leq \underbrace{\sum_{i \in N_{j^*}} y'_i}_{\geq 1} f_i$ $\quad x''_{ij^*}$

iii) $\underbrace{\sum_{i\in I} f_i y''_i}_{\overset{ii)}{\leq} \underbrace{\sum_{i\in I} f_i y'_i}_{\overset{325 ii)}{\leq} 2\sum_{i\in I} f_i \overline{y}_i}} + \underbrace{\sum_{ij\in A} d_{ij} x''_{ij}}_{\overset{i)}{\leq} 6 \sum_{j\in J} \underbrace{D_j}_{= \underbrace{\sum_{i\in I} \overline{x}_{ij} d_{ij}}_{\leq 6 \sum_{ij\in A} \overline{x}_{ij} d_{ij}}}} \leq 6 \sum_{i\in I} f_i \overline{y}_i + \sum_{ij\in A} d_{ij} \overline{x}_{ij}$.

32.7 Thm (Shmoys, Tardos, Aardal [1997]): Alg. 323 = 6-approximate.

32.8 Lemma (generalization of Claim 3.2.4): For $\beta > 0$ let

$N_j(\beta) := \{ i \in I : \overline{x}_{ij} > 0 \wedge d_{ij} \leq \beta D_j \}$.

Then $\sum_{i\in N_j(\beta)} \overline{x}_{ij} \geq 1 - \frac{1}{\beta} = \frac{\beta - 1}{\beta}$.

Proof: Let

$F_j(\beta) := \{ i \in I : \overline{x}_{ij} > 0 \wedge d_{ij} > \beta D_j \}$

$s_j := \sum_{i\in I} \overline{x}_{ij} \geq 1$

Suppose $\sum_{i\in F_j(\beta)} \overline{x}_{ij} > \frac{s_j}{\beta}$. Then

$$
\begin{aligned}
D_j = \sum_{i\in I} d_{ij} \overline{x}_{ij} &= \sum_{i\in F_j(\beta)} d_{ij} \overline{x}_{ij} + \sum_{i\in N_j(\beta)} d_{ij} \overline{x}_{ij} \\
&\geq \sum_{i\in F_j(\beta)} d_{ij} \overline{x}_{ij} \\
&\geq \beta D_j \sum_{i\in F_j(\beta)} \overline{x}_{ij} \\
&> \cancel{\beta} D_j \frac{s_j}{\cancel{\beta}} \\
&\geq D_j \quad \lightning
\end{aligned}
$$

$\Rightarrow s_j = \underbrace{\sum_{i\in F_j(\beta)} \overline{x}_{ij}}_{\leq \frac{s_j}{\beta}} + \sum_{i\in N_j(\beta)} \overline{x}_{ij} \Rightarrow \sum_i \overline{x}_{ij}$

$\Rightarrow \sum_{i\in N_j(\beta)} \overline{x}_{ij} \geq s_j - \frac{s_j}{\beta} = s_j\left(1 - \frac{1}{\beta}\right) \geq 1 - \frac{1}{\beta}$. $\square$