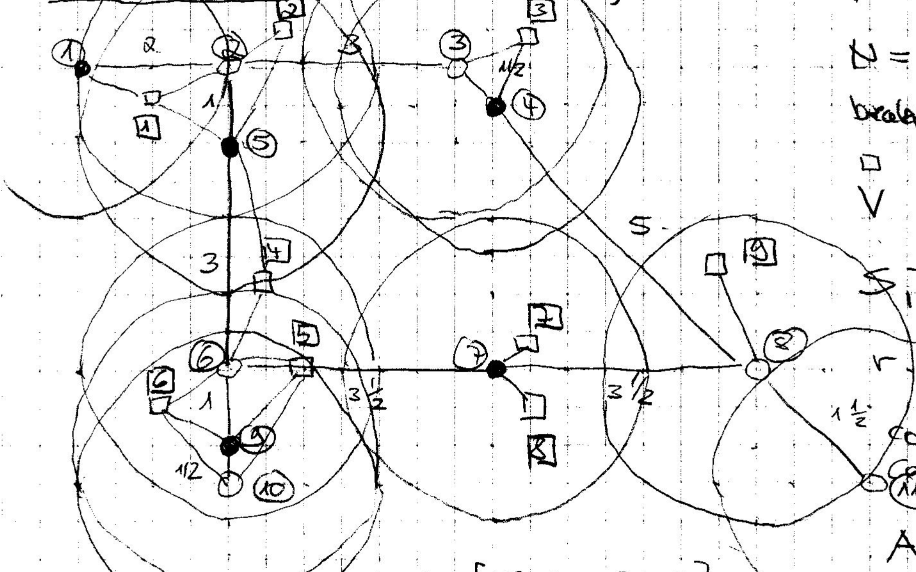
2.5 Stop Location Problems in Networks

2.5.1 Ex. (Schöbel [2005]): Bus stop location problem (BSLP)



$N = (\underbrace{S}_{\text{breakpoints}} \cup \underbrace{T}_{\text{existing stops}}, E)$ bus network

V finite set of demand points

$S, T, V \subseteq \mathbb{R}^2$ planar

$r \in \mathbb{R}_+$ covering radius

$\mathrm{cov}_r(U) := \{v \in V : \|v, U\| \le r\}$ cover of $U \subseteq \mathbb{R}^2$

$\mathrm{cov}_r(u) := \mathrm{cov}_r(\{u\}),\ u \in \mathbb{R}^2$ ~~and acc. finite~~

$A^{\mathrm{cov}}_r(U) := \left(\chi^{\mathrm{cov}_r(u)}\right)_{u \in U}$ covering matrix assoc. with $U \subseteq \mathbb{R}^2$

$U = \{5, 3, 6, 7, 8\}$
$p = |U| = 5$

$S = \{2, 3, 6, 8, 10, 11\}$

$T = \{1, 4, 5, 7, 9\}$

$V = \{1, \ldots, 9\}$

$r = 2$

$$A^{\mathrm{cov}}_2(S \cup T) = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{matrix} \\ \begin{pmatrix} 1 & 1 & & & 1 & & & & & & \\ & 1 & & & 1 & & & & & & \\ & & 1 & 1 & & & & & & & \\ & & & & 1 & 1 & & & & & \\ & & & & & 1 & & & 1 & 1 & \\ & & & & & 1 & & & 1 & 1 & \\ & & & & & & 1 & & & & \\ & & & & & & 1 & & & & \\ & & & & & & & 1 & & & \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

2.5.2. Def. (Discrete bus stop location problems):

a) (DSL) $p = |U| \,/\, S \,/\, \mathrm{cov}_r(U) = V \,/\, l_2 \,/\, p$ planning from scratch

b) (DSL 1) $p = |U| \,/\, T \,/\, \mathrm{cov}_r(U) = V \,/\, l_2 \,/\, p$ closing stops

c) (DSL 2) $p = |U| \,/\, S \,/\, \mathrm{cov}_r(U \cup T) = V \,/\, l_2 \,/\, p$ opening stops

d) (DSL 3) $p = |U| \,/\, S \cup T \,/\, \mathrm{cov}_r(U) = V \,/\, l_2 \,/\, p$ closing & opening stops

2.5.3. Obs.: (DSL i) is a special case of (DSL), $i = 1, 2, 3$.

Proof.: Homework.

2.5.4. Def. (Continuous bus stop location problem).

(CSL) $p = |U| \,/\, N \,/\, \mathrm{cov}_r(U) = V \,/\, l_2 \,/\, p$ planning from scratch

2.5.5. Rem. (Alternative objective functions): One can also consider

b) $p = |U| \,/\, \{V, N\} \,/\, \mathrm{cov}_r(U) \,/\, \le p \,/\, \sum_{ij \in V^2} w_{ij}$ travel time (more difficult)

a) " $/\, l_2 \,/\, \sum w_u$ construction costs

2.6 The Discrete Stop Location Problem

2.6.3 Prop. (Set Covering Model): (DSL) = $p = |U| / S / \mathrm{cov}_r(u) = V / \ell_2 / p$ can be formulated as a set covering problem as follows:

$$\min \mathbb{1}^T x \qquad = \qquad \min \sum x_s$$

$$A^{cov}_r(S)\, x \geq \mathbb{1} \qquad \sum_{v \in \mathrm{cov}_r(S)} x_s \geq 1 \quad \forall v \in V$$

$$x \in \{0,1\}^S \qquad x_s \in \{0,1\} \quad \forall u \in S.$$

2.6.4 Cor. (Complexity of DSL) DSL is NP / APX-hard.

Proof: The set covering problem is NP / APX-hard even for $c = \mathbb{1}$. □

2.6.1. Def. (Set Covering Problem): Let $A \in \{0,1\}^{m \times n}$, $c \in \mathbb{R}^n_+$.

(SCP) $\min c^T x,\ Ax \geq \mathbb{1},\ x \in \{0,1\}^n$

Set covering problem: Any solution x of (SCP) as well as its support, is a cover.

We can assume w.l.o.g.:
a) $A_{i\cdot} \neq 0 \ \forall i$ (SCP is feas.)
b) $c_j > 0 \ \forall j$

2.6.2. Ex. (Set Covering Problem, see Ex. 2.5.1)

$$\min \sum_{i=1}^{11} x_i$$

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 1&2&3&4&5&6&7&8&9&10&11 \end{array}\\
\left(\begin{array}{ccccccccccc}
1&1&&&1&&&&&&\\
&1&&&1&&&&&&\\
&&1&1&&&&&&&\\
&&&&1&1&&&&&\\
&&&&&1&&&1&1&\\
&&&&&1&&&1&1&\\
&&&&&&1&&&&\\
&&&&&&1&&&&\\
&&&&&&&1&&&
\end{array}\right)
\end{array}
\begin{pmatrix} x_1 \\ \vdots \\ x_{11} \end{pmatrix} \geq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$x_i \in \{0,1\},\ i = 1, \dots, 11$$

$\leadsto x_8 = 1$
$x_{11} = 0$
$x_7 = 1$
$x_1 = 0$
$x_2 = 0$
...

see Tut.

2.6.5 Alg. (Greedy Algorithm for the SCP):

Input: $A \in \{0,1\}^{m \times n}$, $c \in \mathbb{R}^n_+$, $c_j > 0 \ \forall j$

Data structures: $A_j = \mathrm{supp}(A_{\cdot j})$, $j = 1, \dots, n$,

Output: Cover J_*.

It. 0. $J_* = \emptyset$.

It. 1. If $A_j = \emptyset \ \forall j$ then stop, output J_*
else $k \leftarrow \operatorname{argmax} |A_j| / c_j$

2. $J_* \leftarrow J_* \cup \{k\}$, $A_j \leftarrow A_j \setminus A_k$, $j = 1, \dots, n$, goto 1.

2.6.6 Rem.: $\operatorname{argmax} |A_j| / c_j = \operatorname{argmin}_{|A_j| \neq 0} c_j / |A_j|$, i.e., Alg. 2.6.5 adds in each iteration a column k to the cover-to-be that minimizes the cost per yet uncovered row.

2.6.7 Ex. (Greedy Alg.)

a) $\min \sum x_j$

$$\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} &
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0
\end{pmatrix}
\end{array}
\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_6 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Iteration 0 : $J_*^i = \emptyset$

" 1 : $k \in \{1,2,5\}$, $k \leftarrow 1$, $J_*^i = \{1\}$

" 2 : $k \in \{3,7\}$, $k \leftarrow 3$, $J_* = \{1,3\}$

" 3 : output $J_*^i = \{1,3\}$.

b) $A_j = \{j\}$, $c_j = 1/j$, $j = 1, \dots, m$, $A_{m+1} = \{1, \dots, m\}$, $c_{m+1} > 1$, i.e.,

$c = 1 \quad \frac{1}{2} \quad \frac{1}{3} \quad \dots \quad \frac{1}{m} \quad \alpha > 1$

$$A = \left(\begin{array}{cccc|c}
1 & & & & 1 \\
 & 1 & & & 1 \\
 & & 1 & & 1 \\
 & & & \ddots & \vdots \\
\hline
 & & & 1 & 1
\end{array}\right)$$

Iteration 0 : $J_* = \emptyset$

" 1 : $k = \arg\max \left\{ \frac{1}{1/1}, \frac{1}{1/2}, \dots, \frac{1}{1/m}, \frac{m}{\alpha} \right\} = m$, $J_* = \{m\}$

" 2 : $k = \arg\max \left\{ \frac{1}{1/1}, \frac{1}{1/2}, \dots, \frac{1}{1/(m-1)}, \frac{m-1}{\alpha} \right\} = m-1$, $J_* \{m, m-1\}$

⋮

" m : $k = \arg\max \left\{ \frac{1}{1/1}, \frac{1}{\alpha} \right\} = 1$, $J_* = \{1, \dots, m\}$

" m+1 ; Output $J_* = \{1, \dots, m\}$

$$c_{J_*^i} = \sum_{j=1}^{m} \frac{1}{j} =: \underbrace{H(m)}_{\text{m-th harmonic number}} > \alpha = c_{J_{opt}}.$$

$\Rightarrow c_{J_*} \geq H(m) \, c_{J_{opt}}$.

2.6.8 Thm (Harmonic Approx. Ration for the SCP, Chvátal [1979])

Alg. 2.6.5 produces a selection J_* such that

$$c_{J_*} \leq \sum_{j \in J_{opt}} H(j) \, c_j \leq H\left(\max |A_{\cdot j}|\right) c_{J_{opt}} \leq H(m) \, c_{J_{opt}}.$$

2.6.9 Obs. : For Ex. 2.6.7 a) we have

$$c_{J_*} = c_{\{1,3\}} = 2 \leq \left(1 + \frac{1}{2} + \frac{1}{3}\right) c_{J_{opt}} = \frac{11}{6} c_{J_{opt}} \Rightarrow c_{J_{opt}} \geq 2 \Rightarrow J_* = J_{opt}$$

Proof of thm 2.6.8.: Consider the following chain of inequalities:

$$\min \sum_{j=1}^{n} H(\underbrace{\sum_{i=1}^{m} a_{ij}}_{=|A_j|}) c_j x_j \overset{\text{LP relaxation}}{\geq} \min \sum_{j=1}^{n} H\Big(\sum_{i=1}^{m} a_{ij}\Big) c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq 1, \quad i=1,\dots,m \qquad \sum_{j=1}^{n} a_{ij} x_j \geq 1, \quad i=1,\dots,m$$

$$x_j \in \{0,1\} \qquad x_j \geq 0, \quad j=1,\dots,n$$

$$\overset{\text{LP duality}}{\geq} \max \sum_{i=1}^{m} y_i$$

$$(1) \quad \sum_{i=1}^{m} a_{ij} y_i \leq H\Big(\sum_{i=1}^{m} a_{ij}\Big) c_j, \quad j=1,\dots,n$$

$$(2) \quad y_i \geq 0, \quad i=1,\dots,m$$

$$(3) \quad \geq c_{J_*}.$$

If (1) holds, the claim follows by setting $x = x^{J_{opt}}$. Denote

a) A_j^r the set A_j at the beginning of iteration r

b) $w_j^r := |A_j^r|$

c) J_*^r the set J_* " and " r, assume w.l.o.g. $J_*^r = \{1,\dots,r\}$.

d) $y_i := c_r / w_r^r$, $i \in A_r^r$, i.e. $y_i \triangleq$ price to cover row i

e) t number of iterations in which J_* grows.

Then:

i) $w_r^r / c_r \geq w_j^r / c_j$, $r=1,\dots,t$, $j=1,\dots,n$

ii) $c_{J_*} = \sum_{j=1}^{t} c_j$

We claim that y as defined in d) satisfies (1), (2), and (3).

(2): ✓

(3): $$\sum_{i=1}^{m} y_i = \sum_{r=1}^{t} \Big(\sum_{i \in A_r^r} \overbrace{y_i}^{=c_r/w_r^r}\Big) = \sum_{r=1}^{t} w_r^r \cdot c_r / w_r^r = \sum_{r=1}^{t} c_r = c_{J_*}.$$

(1): $$\sum_{i=1}^{m} a_{ij} y_i = \sum_{r=1}^{t} \sum_{\underbrace{i \in A_j \cap A_r^r}_{= A_j^r \setminus A_j^{r+1}}} y_i = \sum_{r=1}^{t} (w_j^r - w_j^{r+1})(c_r / w_r^r)$$

$$s = \operatorname{argmax}_{r=1,\dots,t}\{w_j^r > 0\} \leq \sum_{r=1}^{s} (w_j^r - w_j^{r+1}) \overbrace{(c_r / w_r^r)}^{\overset{(ii)}{\leq} c_j / w_j^r}$$

$$\leq c_j \sum_{r=1}^{s} (w_j^r - w_j^{r+1}) / w_j^r$$

$$\overset{(Ex)}{\leq} c_j \sum_{r=1}^{s} \big[H(w_j^r) - H(w_j^{r+1})\big]$$

$$\leq H(w_j^1)\, c_j, \quad j=1,\dots,n \qquad \square$$

$$\frac{9-7}{9} + \frac{7-5}{7} + \frac{5-3}{5} + \dots$$
$$\frac{2}{9} + \frac{2}{7} + \frac{2}{5} + \dots$$
$$\leq \frac{1}{9}+\frac{1}{8} + \frac{1}{7}+\frac{1}{6} + \frac{1}{5}+\frac{1}{4}$$
$$\frac{n-k}{n} = 1 - \frac{k}{n} \leq H(n) - H(k) = \frac{1}{n} + \dots + \frac{1}{k+1}$$

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