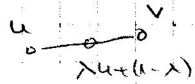
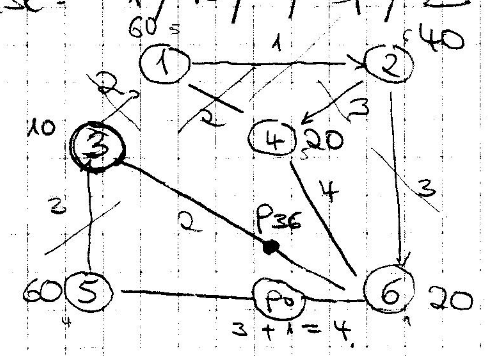
2.3 Medians in Networks



2.3.1. Ex. (Schöbel & Schmidt [2009]): shortest path distance w.r.t. N, $sp(\lambda u+(1-\lambda)v, w) := \min\{(1-\lambda)c_{uv} + sp(u,w),\ \lambda c_{uv} + sp(v,w)\}$

Warehouse: $\underbrace{1}_{\in V\cup A\cup E} / N / \cdot / \underbrace{sp}_{\text{shortest path distance}} / \Sigma$



$\underbrace{\text{undirected}}\ \underbrace{\text{directed}}$ edges

$N = (V, E\cup A)$ network

$c_e, c_a \in \mathbb{R}_+$ edge weights

i) $p=4$: $\sum_{v\in V} sp(4,v) = 2+3+6+0+8+4 = 23$

ii) $p=p_0$: $\sum_{v\in V} sp(p_0,v) = 5+6+3+5+3+1 = 23$

2.3.2. Thm. (Node solution, Hakimi [1964]): There is $v\in V$ s.t.

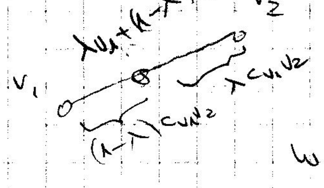
$v \in \operatorname{argmin} 1/N/\cdot/sp/\Sigma$,

i.e., the 1-median problem on a network has an optimal node solution.

Proof: Consider $p = \lambda v_1 + (1-\lambda)v_2$, $\lambda\in(0,1)$ for:

a) $v_1v_2 \in A$: $\sum_{v\in V} sp(p,v) > \sum_{v\in V} sp(v_2,v)$.

b) $v_1v_2\in E$: $\sum_{v\in V} sp(p,v) = \sum_{v\in V} \min\{\underbrace{(1-\lambda)c_{v_1v_2} + sp(v_1,v)}_{:\Leftrightarrow v\in V_1},\ \underbrace{\lambda c_{v_1v_2} + sp(v_2,v)}_{\Leftrightarrow: v\in V_2}\}$

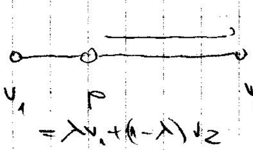


$$= \sum_{v\in V_1}\left[(1-\lambda)c_{v_1v_2} + sp(v_1,v)\right] + \sum_{v\in V_2}\left[\lambda c_{v_1v_2} + sp(v_2,v)\right]$$

$$= c_{v_1v_2}\left[(1-\lambda)|V_1| + \lambda|V_2|\right] + \sum_{v\in V_1} sp(v_1,v) + \sum_{v\in V_2} sp(v_2,v)$$

$$\overset{\substack{\text{w.l.o.g.}\\ |V_1|\le|V_2|}}{\geq} c_{v_1v_2}\left[1\cdot|V_1| + 0\,|V_2|\right] + \sum_{v\in V_1} sp(v_1,v) + \sum_{v\in V_2} sp(v_2,v)$$

$$\overset{(p\to v_2)}{=} \sum_{v\in V_1}\underbrace{\left[c_{v_1v_2} + sp(v_1,v)\right]}_{\geq sp(v_2,v)} + \sum_{v\in V_2} sp(v_2,v)$$



$$\geq \sum_{v\in V} sp(v_2,v).$$ □

2.3.3. Alg. (for $1/N/\cdot/sp/\Sigma$)

Input: $N=(V,E\cup A)$, $c_w\in\mathbb{R}_+$, $w\in E\cup A$

Output: p^* - argmin $1/N/\cdot/sp/\Sigma$

1. Compute shortest path matrix $(sp(u,v))_{uv\in V^2}$ (e.g., Floyd-Warshall)
2. compute row sums $sp^T \mathbb{1}$
3. output index of row sum minimum $p^* = \operatorname{argmin} sp^T\mathbb{1}$. □

2.3.1 Ex (cont'd):

$$Sp = \begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
1 & 0 & 1 & 6 & 2 & 8 & 4 \\
2 & 5 & 0 & 5 & 3 & 7 & 3 \\
3 & 2 & 3 & 0 & 4 & 6 & 2 \\
4 & 2 & 3 & 6 & 0 & 8 & 4 \\
5 & 5 & 6 & 3 & 7 & 0 & 4 \\
6 & 4 & 5 & 2 & 4 & 4 & 0
\end{array} \quad \begin{array}{c} \\ 21 \\ 23 \\ 17 \\ 23 \\ 25 \\ 19 \end{array} \Rightarrow p^* = 3.$$

2.3.4. Ex (Weighted median in a network):

Volumes w_i: $1/N|\cdot|sp/\sum w_i$

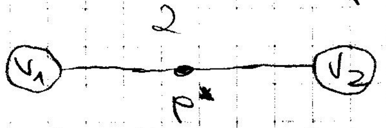
$w_i = (60, 40, 10, 20, 60, 20) \Rightarrow p^* = \operatorname{argmin} sp^T w$.

$$\begin{pmatrix} 0&1&6&2&8&4 \\ 5&0&5&3&7&3 \\ 2&3&0&4&6&2 \\ 2&3&6&0&8&4 \\ 5&6&3&7&0&4 \\ 4&5&2&4&4&0 \end{pmatrix} \begin{pmatrix} 60 \\ 40 \\ 10 \\ 20 \\ 60 \\ 20 \end{pmatrix} = \begin{pmatrix} 700 \\ 890 \\ 720 \\ 260 \\ 790 \\ 780 \end{pmatrix} \Rightarrow p^* = 1.$$

2.4 Centers in Networks

2.3.4 Ex. (cont'd): Fire brigade $1/N|\cdot|sp/\max$

2.4.1. Ex. (Edge selection):



$\operatorname{argmin} 1/N|\cdot|sp/\max = \frac{1}{2} v_1 + \frac{1}{2} v_2$.

2.4.2 Obs.: Center problems in networks do not necessarily have optimal node selections

allowing only node selections

2.4.3 Alg. (for $1/N|N|\cdot|sp/\max$)

Input: $N = (V, E \cup A)$, $c_{uv} \in \mathbb{R}_+$, $uv \in E \cup$

Output: $p^* = \operatorname{argmin} 1/N|V|\cdot|sp/\max$

1. compute $(sp(u,v))_{uv \in V^2}$
2. compute row maxima $\max_{v \in V} (sp(u,v))$
3. output index of minimal row max. $p^* = \operatorname{argmin} \max_v sp(u,v)$ $\square$

2.3.4 Ex. (cont'd):

$$\begin{pmatrix} 0&1&6&2&8&4 \\ 5&0&5&3&7&3 \\ 2&3&0&4&6&2 \\ 2&3&6&0&8&4 \\ 5&6&3&7&0&4 \\ 4&5&2&4&4&0 \end{pmatrix} \quad \begin{array}{c} 8 \\ 7 \\ 6 \\ 8 \\ 7 \\ 5 \end{array} \Rightarrow p^* = 6, \quad \max_{v \in V} sp(6,v) = 5$$

2.4.4 Lem. (Preprocessing): Let $p' = \lambda v_1 + (1-\lambda) v_2$, $\lambda \in (0,1)$, $v_1 v_2 \in E \cup A$ and $\bar{p} = \operatorname{argmin} 1/N|V|/\cdot/sp/\max$. Then $p' \notin \operatorname{argmin} 1/N|\cdot|/sp/\max$ if

a) $v_1 v_2 \in A$ and $c_{v_1 v_2} > 0$

b) $v_1 v_2 \in E$ and $sp(v_1, \bar{v}), sp(v_2, \bar{v}) \geq \max_{v \in V} sp(\bar{p}, v)$ for some $\bar{v} \in V$.

Proof:

a) $\max_{v \in V} sp(p, v) > \max_{v \in V} sp(v_2, v)$.

b) $\max_{v \in V} sp(p, v) \geq sp(p, \bar{v}) > \min_{i=1,2} sp(v_i, \bar{v}) \geq \max_{v \in V} sp(\bar{p}, v)$. □

2.3.4. Ex (cont'd): $14 \in E$, $p \in 14$

$sp(p,5) \geq \min\{sp(1,5), sp(4,5)\} = \min\{8,8\} = 8 \geq 5 = \max_{v \in V} sp(\overset{\bar{p}}{\overset{\shortparallel}{6}}, v)$

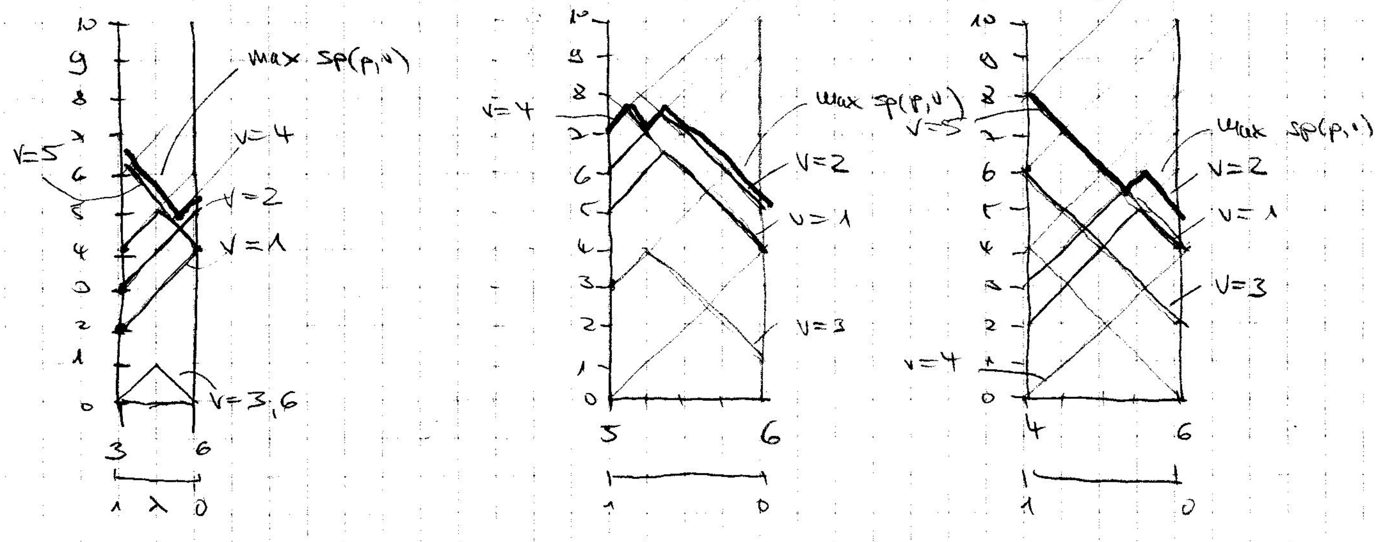
2.4.5 Obs. (Undirected edges): Let $p = \lambda v_1 + (1-\lambda) v_2$, $\lambda \in (0,1)$, $v_1 v_2 \in E$

a) $sp(p,v) = \min\{(1-\lambda) c_{v_1 v_2} + sp(v_1, v), \lambda c_{v_1 v_2} + sp(v_2, v)\}$, $v \in V$

is a piecewise affine, continuous function in λ with ≤ 2 pieces

b) $\max sp(p,v)$ is a piecewise affine, continuous function in λ with $\leq 2|V|$ pieces.

2.3.4. Ex. (cont'd)



$\Rightarrow \lambda = 0{,}25$ $\quad\quad \rightarrow \lambda = 0$ $\quad\quad \rightarrow \lambda = 0$

$\Rightarrow p_{36} = \frac{1}{4} ③ + \frac{3}{4} ⑥$ $\quad\quad \rightarrow p_{56} = ⑥$ $\quad\quad \rightarrow p_{46} = 6$

$\max_{v \in V} sp(p_{36}, v) = \frac{9}{2}$ $\quad\quad \max_{v \in V} sp(6, v) = 5$ $\quad\quad \max_{v \in V} sp(6, v) = 5$

$6 = \operatorname{argmin} 1/N|V|/\cdot/sp/\max$

$\Rightarrow p^* = \operatorname{argmin} 1/N|\cdot/sp/\max = \frac{1}{4} ③ + \frac{3}{4} ⑥ = p_{36}$.

21.05.12