Technische Universität Berlin
Institut für Mathematik

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Mathematics of Infrastructure Planning

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Exercise sheet 3

Deadline: Thu, Mai 03, 2012, **23:59**, `mailto:borndoerfer@zib.de`

Exercise 3. **10 points**

Consider a set of points $V \subseteq \mathbb{R}^2$ in the plane and let $p^* = \operatorname{argmin} 1/\mathbb{R}^2/\cdot/\ell_2^2/\sum$ be their median w.r.t. squared Euclidean distances. Prove that $p^* \in \mathsf{conv}\, V$.

Exercise 4. **10 points**

Consider a triangle $\Delta = \mathsf{conv}\{v_1, v_2, p\}$ in $\mathbb{R}^2$ and let $c = (v_1 + v_2)/2$ be the median of v_1 and v_2. Prove that $||p - c||_2 \leq (||p - v_1||_2 + ||p - v_2||_2)/2$.

Exercise 5. **10 points**

Weiszfeld's algorithm iterates the operator T defined as

$$p_{j+1} := T(p_j) := \frac{1}{\sum_{i=1}^{m} \frac{1}{||p_j - v_i||_2}} \sum_{i=1}^{m} \frac{v_i}{||p_j - v_i||_2}, \quad j = 1, 2, \ldots$$

to determine the median $p^* = \operatorname{argmin} 1/\mathbb{R}^2/\cdot/\ell_2/\sum$ of a given set of points $V = \{v_i\}_{i=1}^m$ in $\mathbb{R}^2$ w.r.t. Euclidean distances. Consider $V = \{(-1, 3), (0, 0), (9, 9), (10, 0)\}$ and compute the median p^* numerically. Try $p_0 = (8, -1)$ and two other starting points of your choice.

Exercise 6. **(Tutorial session)**

Consider the 1-median problem $1/\mathbb{R}^2/\cdot/\ell_1/\sum$ w.r.t. Manhattan distances for a set of points $V \subseteq \mathbb{Z}^2$ with integer coordinates. Use Fig. 1 to construct an instance of this problem with at least 6 different points s.t.

1. the set of medians is a line segment.
2. the set of medians is a single point.

Exercise 7. **(Tutorial session)**

Consider the 1-center problem $1/\mathbb{R}^2/\cdot/\ell_2/\max$ w.r.t. Euclidean distances for a set of points $V = \{v_i\}_{i=1}^m \subseteq \mathbb{R}^2$ in the plane.

a) What is the median for $m = 2$?
b) What is the median for $m = 3$ if $\mathsf{conv}\{v_1, v_2, v_3\}$ has an obtuse or right angle?
c) What is the median for $m = 3$ if $\mathsf{conv}\{v_1, v_2, v_3\}$ has all acute angles?

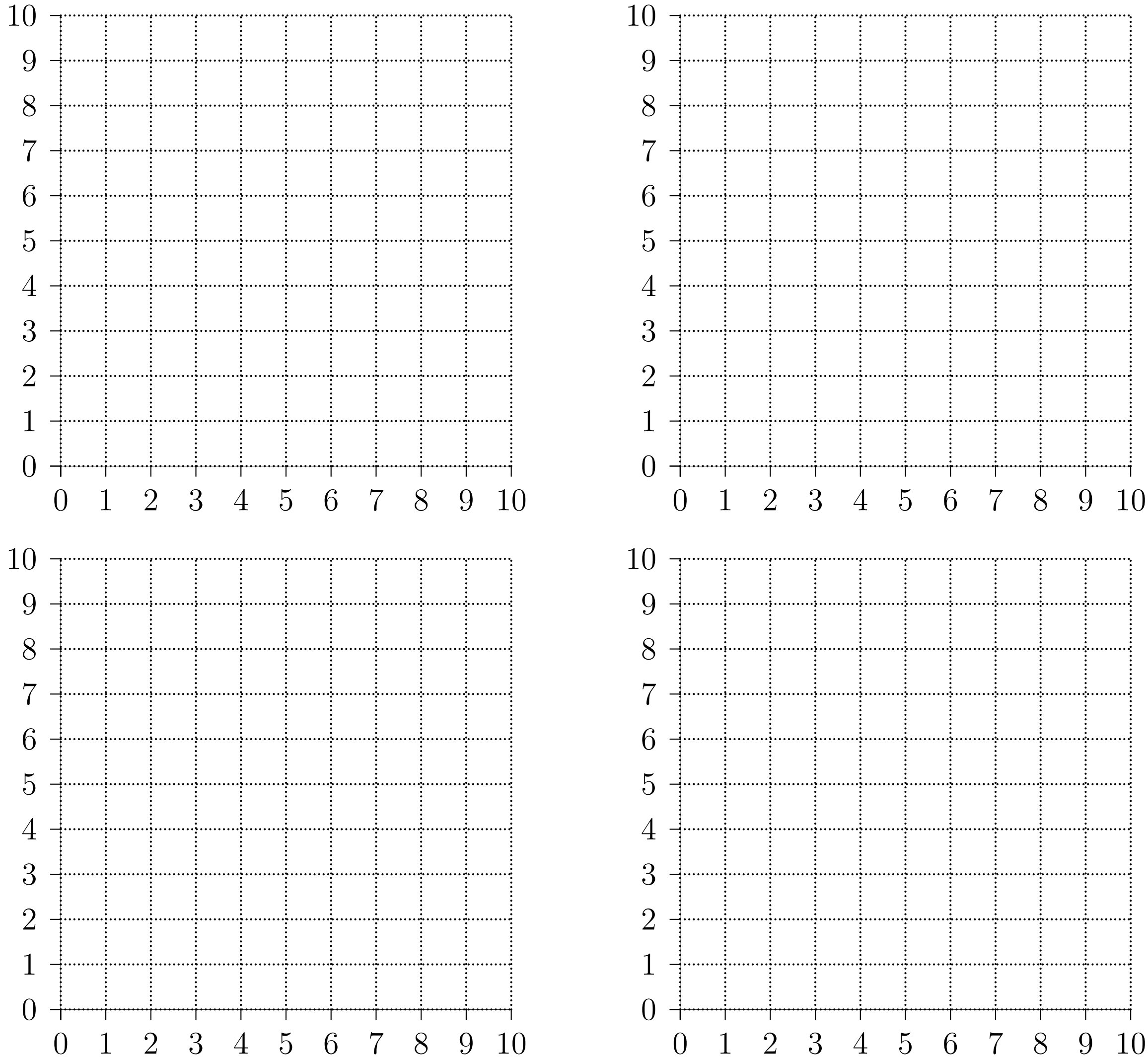

Figure 1: 1-median ℓ_1-problem.

d) The case $m > 3$ can be reduced to a)–c) by considering all 2- and 3-tuples of points. Using this fact and Fig. 2, solve the 4-point instance given by $V = \{(2,0), (2,8), (6,3), (8,2)\}$ graphically.

Exercise 8. **(Tutorial session)**

Consider the 6-node graph $N = (V, E)$ in Fig. 3 with distances d_{ij} and demands w_i as drawn next to the edges and nodes.

1. Solve the warehouse location problem $1/V/\cdot/d_{ij}/\sum w_i$.
2. Solve the warehouse location problem $2/V/\cdot/d_{ij}/\sum w_i$ by fixing the solution of a) and adding a second warehouse in a best possible way.
3. Develop an IP formulation for $2/V/\cdot/\text{shortest path}/\sum w_i$.
4. Solve your formulation from c).
5. Did b) produce the optimum?

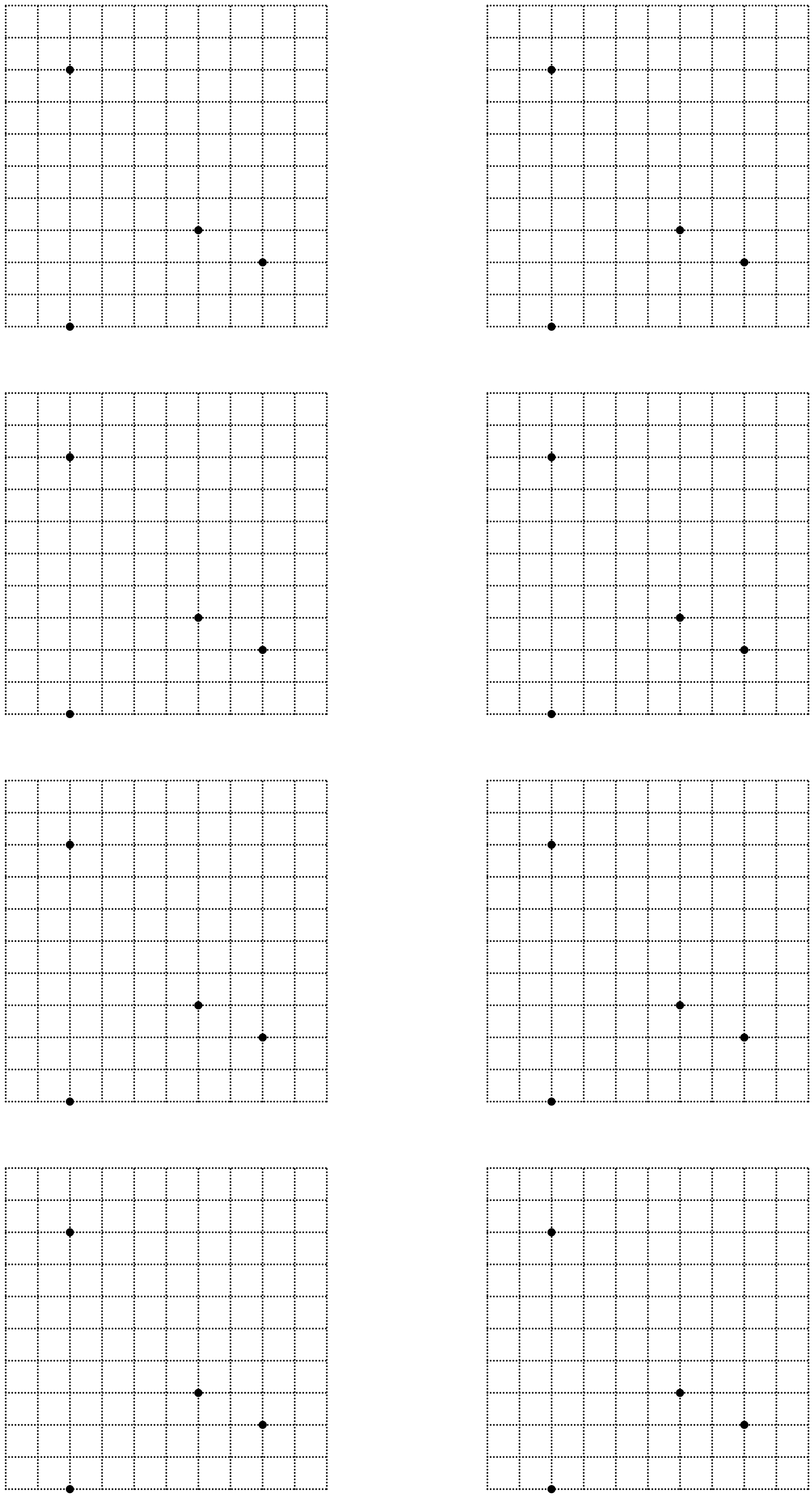

Figure 2: 1-center ℓ_2-problem.

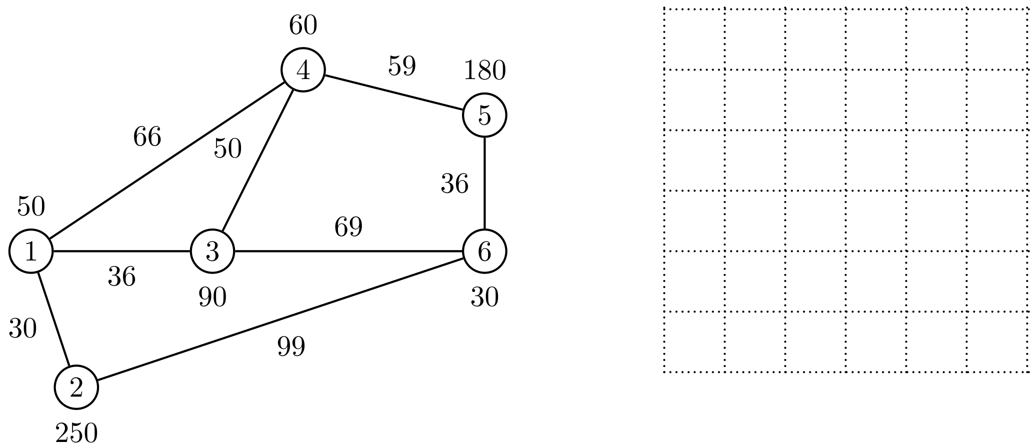

Figure 3: Warehouse location problem.