Employing Mixed-Integer Rounding in Telecommunication Network Design

Diplomarbeit bei
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Hiermit versichere ich an Eides statt die selbständige
und eigenhändige Anfertigung dieser Diplomarbeit.

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Zusammenfassung

In dieser Diplomarbeit werden grundlegende Probleme der kostenoptimalen Dimensionierung von Telekommunikationsnetzwerken untersucht. Diese werden als lineare gemischt ganzzahlige Programme formuliert, wobei sich in der Modellierung auf die Konzepte *Routing* und *Kapazitätszuweisung* beschränkt wird. Es werden parallel drei übliche, aus der Praxis motivierte Möglichkeiten behandelt, die auf gerichteten oder ungerichteten Kanten eines Netzwerkes installierte Kapazität zu nutzen. Diese unterscheiden wir als *DIrected*, *BIdirected* und *UNdirected*. Die studierten Probleme treten als Relaxierungen vieler realistischer Fragestellungen der Netzwerkoptimierung auf. Sie enthalten elementare Strukturen, deren Studium ausschlaggebend ist für das Verständnis komplexerer Modelle. Letztere können zusätzliche Erfordernisse berücksichtigen, wie zum Beispiel die Ausfallsicherheit von Netzwerken.

Zur Lösung solcher $\mathcal{NP}$-schweren Optimierungsprobleme werden erfolgreich Branch & Bound und Schnittebenenverfahren kombiniert (Branch & Cut). Für die Wirksamkeit dieser Algorithmen ist es sehr nützlich, möglichst genaue Kenntnisse der Struktur der Seitenflächen der zugrundeliegenden Polyeder zu haben, welche die konvexe Hülle der Lösungsmenge beschreiben. Es sind starke gültige Ungleichungen zu finden, welche hochdimensionale Seitenflächen oder sogar Facetten definieren. Diese sollten zudem schnell separiert werden können und die numerische Stabilität der Algorithmen möglichst nicht beeinflussen.

Diese Arbeit beschäftigt sich im Wesentlichen mit der sehr allgemeinen Rundungstechnik *Mixed-Integer Rounding* (*MIR*) zur Verstärkung gültiger Ungleichungen unter Verwendung der Ganzzahligkeitsnebenbedingungen. Es wird eine *MIR*-Prozedur motiviert, bestehend aus den Schritten *Aggregieren*, *Substituieren*, *Komplementieren* und *Skalieren*, welche durch Ausnutzung der Struktur der gegebenen Parameter zu einer gültigen Basisungleichung führt, die dann durch *MIR* eine starke und oft facetten-induzierende Ungleichung gibt. Es werden verschieden Klassen solcher Ungleichungen untersucht und auf ihre Praxistauglichkeit beim Einsatz in Branch & Cut-Verfahren getestet.

Nach einer kurzen Einführung werden in Kapitel 2 die für uns in dieser Diplomarbeit relevanten Probleme definiert. Kapitel 3 gibt eine ausführliche Übersicht über die Technik *MIR*. Wir beschäftigen uns vor allen Dingen mit den Begriffen *Superadditivität* und *Lifting* und behandeln Aspekte wie Numerik und beschränkte Variablen.

Kapitel 4 und Kapitel 5 umfassen Untersuchungen zu so genannten *cut sets*. Diese Polyeder werden durch Schnitte in Netzwerken definiert und relaxieren die von uns behandelten Probleme. Hauptsächlich durch *MIR* entwickeln wir sowohl neue als auch bekannte Klassen von facetten-definierenden Ungleichungen für *cut sets*, wobei strukturelle Unterschiede herausgearbeitet werden, die durch die drei verschiedenen Typen der Kapzitätsbereitstellung und durch beschränkte Variablen ent-

stehen. Als ein zentrales Resultat wird bewiesen unter welchen Bedingungen facetten-induzierende Ungleichungen für cut sets auch Facetten der zugehörigen relaxierten Polyeder sind.

Im Kapitel 6 geben wir weitere Typen von *MIR*-Ungleichungen an, die auf anderen Netzwerkstrukturen basieren und weisen ferner auf offene Fragen sowie interessante Ideen hin.

Das Kapitel 7 widmet sich schließlich der Entwicklung und Implementation von Separationsalgorithmen. Wir testen einige der entwickelten Ungleichungsklassen mit Hinblick auf Ihre Wirksamkeit zur Lösung von realistischen Problemen der Netzwerkdimensionierung aus der Telekommunikation und diskutieren die Ergebnisse.

Abstract

In this thesis some basic mixed integer programming models for the design of telecommunication networks are investigated. These models cover bifurcated routing and modular capacity assignment. Three common types of capacity usage are distinguished and bounded as well as unbounded link design variables are considered. This work focuses on the use of Mixed-Integer Rounding (*MIR*) to strengthen the initial problem formulations. A general *MIR*-procedure (based on Marchand & Wolsey [1998], Louveaux & Wolsey [2003]) is applied to the corresponding network design polyhedra that, by exploiting the structure of the given parameters such as underlying graphs, capacities and bound constraints, is able to detect different classes of strong valid inequalities. Moreover, the use of *MIR* as a valid superadditive lifting function is emphasised. Several classes of facet-defining *MIR*-inequalities are presented.

Large parts of this thesis address the investigation of polyhedra based on cuts of the network. It is shown under which conditions facet-defining inequalities for these relaxations are facet-defining for the corresponding network design polyhedra. Facet proofs for two new classes of cut set inequalities are provided. Some of the developed *MIR*-inequalities are used as cutting planes within a Branch & Cut algorithm and tested against real-life networks with excellent results.

Contents

Chapter 1

Introduction

Nowadays, global economy depends on high quality data communications as much as on physical transport and businesses increasingly reliant on low cost for their telecommunication needs. The liberalisation of the European telecommunication markets in the last few years, the rapid development of Internet technologies and the demand for new multi-media services puts pressure on telecommunication companies and makes the market more competitive. The major burdens of network carriers and telecommunication service providers are their expenditures for network construction and the costs of bandwidth lease. They generally hope to realise easy to manage networks at low cost that survive certain failure situations and that satisfy all given customer demands.

In the mathematical literature there is a vast variety of approaches to model and solve telecommunication network design problems depending on the requirements to incorporate. All these approaches have two basic concepts in common on which we will concentrate in this thesis. These are *routing* and *capacity assignment*.

We address a network design problem as follows: Given a communication demand between certain locations in a region, the topology of a network connecting these locations has to be determined. All physical links have to be dimensioned by *assigning capacity* such that all demands can be *routed* over the network and the installation cost for capacity is minimal. In practice the possible capacities (bandwidths) always have a discrete structure. We restrict them to a finite set of base units, which may be installed several times on every link of the network (*modular link capacities*). This can be formulated as the problem of minimising a cost function (which is assumed to be linear) over the set $X \subset \mathbb{R}^n$ of all feasible routings and capacity assignments:

$$\min\{\, \kappa^T x : \; x \in X \,\}.$$

Because of the discrete structure of the capacities, some of the variables in x are restricted to integer values. Hence X is a *mixed integer set* and network design optimisation problems as considered in this thesis are *mixed integer programs*.

Despite the absence of more sophisticated requirements as, for instance, survivability of the network, these mixed integer programming problems are known to be $\mathcal{NP}$-hard, meaning that in the sense of complexity theory there exists no efficient (polynomial-time) algorithm to solve them, unless $\mathcal{P} = \mathcal{NP}$.

A common approach to solving hard mixed integer programs is the application of heuristics to find good solutions. However, it is usually impossible to verify their quality. Another possibility, that we will make use of, is an algorithmic framework based on *Branch & Cut* (Wolsey [1998]). Such algorithms provide feasible solutions as well as a quality certificate by computing a lower bound on the cost of the optimal solution. In addition, any primal heuristic can be integrated into a Branch & Cut algorithm.

A very important step in the design of a Branch & Cut algorithm, and crucial for its efficency, is to detect classes of strong valid inequalities for the mixed integer set X. Among other techniques, the linear programming relaxation of the problem is tightened by adding some of these valid inequalities (cutting planes) to the formulation, resulting in a better approximation of the convex hull of X and hence better lower bounds. In fact, a Branch & Cut algorithm largely depends on the quality of the added cutting planes.

One way to generate such inequalities is to exploit the structure of the combinatorial optimisation problem as well as the polyhedral structure of the problem formulation. Very often an analysis of the problem regarding all requirements and constraints turns out to be too complex, and one concentrates on relaxations and simpler structures that somehow reflect certain attributes of the actual problem such as *knapsack sets* and *single node flow sets*.

Another possibility is to consider so-called general purpose cutting planes that do not require any special knowledge about the (combinatorial or polyhedral) structure of the problem. In this category fall *disjunctive*, *split*, *lift and project*, *Chvátal-Gomory*, *Gomory fractional* and *Gomory mixed integer* cuts.

A useful observation in this context is, that many of these general purpose cuts can be derived with the same technique, called *Mixed-Integer Rounding*, which is based on rounding by exploiting given integer constraints. Moreover, recent work documents that some strong valid inequalities detected by problem specific combinatorial and polyhedral studies are in fact *MIR*-inequalities. This is for instance true for classes of *knapsack cover* and *flow cover* inequalities (Chapter 3, 5).

Edmond's blossom inequalities may serve as a simple example for strong inequalities that arise from the polyhedral study of a combinatorial optimisation problem:

$$x(E[S]) \leq \lfloor \tfrac{1}{2}|S| \rfloor,$$

where $G = (V, E)$ is an undirected graph and $S \subseteq V$ a subset of the nodes. These inequalities completely describe the matching polytope together with non-negativity constraints and the inequalities $x(\delta(v)) \leq 1, v \in V$ (Schrijver [2003]). They can be facet-defining for the matching polytope if S contains an odd number of nodes. The validity of the blossom inequalities follows from the combinatorial structure of the matching problem. But without the knowledge of the problem itself we may simply sum up $x(\delta(v)) \leq 1$ for all v in S resulting in the valid *base inequality*

$$2x(E[S]) + x(\delta(S)) \leq |S|.$$

Dividing by 2 and applying *MIR* (Chapter 3) gives a blossom inequality. (In this case the latter is equivalent to applying a Chvátal-Gomory step.).

In this thesis we will follow a mixture of the two mentioned approaches to derive strong valid inequalities for mixed integer sets. We will first exploit knowledge of the structure of network design problems or simpler sets and relaxations to derive promising base inequalities, which will then be strengthened using the general purpose concept *MIR*.

For the last little example we have already used two basic techniques that are concerned with *MIR*, the *Aggregation* of valid inequalities and the *Scaling* of base inequalities. In the subsequent chapters we will show that these techniques can be extended to a more general and sophisticated *MIR* procedure that is able to detect strong valid inequalities for network design polyhedra. For some of them we will investigate if they are even facet-defining, proving the power of *MIR*.

Before giving a thorough definition of the network design problems considered in this thesis in Chapter 2, we will give a short literature review, followed by an outline and some mathematical preliminaries.

1.1 Literature Review

Important literature is referred to at the beginning of each chapter and whenever introducing new theory. But some articles deserve mention here because they are a basic motivation for this thesis.

Marchand [1997] and Marchand & Wolsey [1998] show that known classes of strong valid inequalities for various types of mixed integer sets are in fact *MIR* inequalities. They motivate a generic *MIR* procedure and prove their practical usefulness. In a subsequent paper Louveaux & Wolsey [2003] consider certain cut sets (or single node flow sets), which can be seen as relaxations of network design polyhedra, and make use of a similar *MIR* procedure to develop flow cover and flow pack (reverse flow cover) inequalities.

The polyhedral study of telecommunication network design problems as considered in this thesis was set up by Magnanti & Mirchandani [1993], Magnanti et al. [1993, 1995] and Bienstock & Günlük [1996]. All of their facet-defining inequalities can be obtained by *MIR*.

In a recent article, Atamtürk [2002] is the first to present a detailed polyhedral analysis for cut set polyhedra with unbounded design variables. He provides a complete description of cut sets in the single-commodity, single-facility case and shows how to exactly lift a general class of so- called flow cut inequalities if more than one facility is given. All inequalities as well as the lifting process are based on *MIR*.

In this thesis we bring together some of these different theoretical aspects and put them into the general context of telecommunication network design.

1.2 Outline of the thesis

The thesis is organised as follows: The preliminaries, following this outline, serve as a short reference to the notation and concepts used. Some areas of graph theory and polyhedral theory are covered.

In Chapter 2 we briefly describe telecommunication capacitated network design problems and show how our models can be classified within the existing literature. We introduce the necessary notation and formulate linear mixed integer programming problems for three types of capacity usage.

Chapter 3 investigates *Mixed-Integer Rounding* as a general tool to develop strong valid inequalities for mixed integer sets. We develop notation and terminology used in the subsequent chapters to study certain relaxations of network design problems. Aspects of numerics, superadditivity, lifting and bounded variables are considered.

Chapter 4 focuses on the polyhedral study of cut sets with unbounded design variables. A coherent presentation of the knowledge of these polyhedra and strong valid inequalities is provided. Mainly by applying *MIR*, we present several new classes of facet-defining inequalities, elaborating the differences caused by the three different types of capacity constraints. As a central conclusion, it is demonstrated under which conditions facet-defining inequalities for cut sets are facet-defining for the corresponding network design polyhedra.

In Chapter 5 we again consider cut sets, but with bounded design variables. It is shown how to extend the *MIR* procedures of the preceding chapter in order to exploit the bound constraints by using the observations of Chapter 3. It is proven that so-called flow cover inequalities can be seen as a generalisation of the flow cut inequalities introduced in Chapter 4.

Chapter 6 provides some more examples of strong valid inequalities for network design polyhedra that can be obtained by *MIR* and some open questions and promising ideas are posed.

In Chapter 7 we address the separation problem for some classes of valid inequalities considered in this thesis. Separation heuristics are developed and it is shown how to integrate them into state-of-the-art MIP-solvers. The resulting algorithmic framework is tested on a set of real-life telecommunication networks and we discuss the usefulness of the investigated inequalities.

1.3 Preliminaries

Basic knowledge of graph theory, polyhedral theory, linear optimisation and mixed integer programming is assumed. It was attempted to adhere to standards and especially to Grötschel et al. [1988]. Concerning the fundamental theory the reader is referred to Nemhauser & Wolsey [1988], Wolsey [1998] and Schrijver [2003]. This section only presents some notation and terminology that is used frequently or that diverges from the standard.

1.3.1 Basic notation

If $\mathbb{K}$ is one of the sets $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$, then $\mathbb{K}_+ := \{ x \in \mathbb{K} : \ x \geq 0 \}$. If we want to exclude zero, we explicitely write $\mathbb{K}_+ \backslash \{0\}$. The transposition of a vector $x \in \mathbb{R}^n$ is x^T. So the inner product of two vectors $x, y \in \mathbb{R}^n$ is $x^T y$. The superscript t is solely used for *technologies* or *facilities* as defined in Section 2.2.1. The inequality $x \leq y$ for two vectors x and y is meant to hold component-wise.

If N is a finite index set, then $x \in \mathbb{R}^N$ is a real vector, whose components are indexed by the elements of N. To shorten the notation for sums we write $x(N) := \sum_{i \in N} x_i$. Given $R \subseteq N$, the (sub)vector $x_R \in \mathbb{R}^R$ contains all entries of x that are indexed by elements of R.

For the concept of *Mixed-Integer Rounding* introduced in Chapter 3 some special notation is needed that will be used throughout the thesis. Let $a \in \mathbb{R}$ and $c \in \mathbb{R}_+$. We define:

- $a^+ := \max(0, a)$ and $a^- := \min(0, a)$
- $\langle a \rangle := a - \lfloor a \rfloor$, the fractional part of a
- $r(a, c) := a - c(\lceil \frac{a}{c} \rceil - 1) = \begin{cases} \langle \frac{a}{c} \rangle c & \text{if } \langle \frac{a}{c} \rangle > 0 \\ c & \text{else} \end{cases}$

  Hence $r(a, c)$ is the remainder of a divided by c if $\frac{a}{c} \notin \mathbb{Z}$ and c else. It follows that $r(a, c) \leq c$. Moreover, in the common case that both a and c are integer, $r(a, c) \in \{1, ..., \min(a, c)\} \subset \mathbb{Z}_+$.
- $r(a) := r(a, 1) = a - \lceil a \rceil + 1 = \begin{cases} \langle a \rangle & \text{if } \langle a \rangle > 0 \\ 1 & \text{else} \end{cases}$

1.3.2 Graphs

An undirected **graph** $G = (V, E)$ consists of a set V of **nodes** and a set E of **edges**. Both sets are considered to be non-empty and finite. We will not explicitely use an incidence function $\psi : E \to V \times V$ but will associate an unordered pair of nodes with every edge, called its **endnodes**. We explicitely allow parallel edges. An edge $e \in E$, having the endnodes u and v, is denoted by $e = uv$ only if there is no danger of confusion. No loops are allowed, i. e. the two endnodes of an edge are always distinct.

A directed graph, or **digraph** $G = (V, A)$ consists of a set V of nodes and a set A of **arcs**. With every arc $a \in A$ an ordered pair of (end)nodes (u, v), with $u \neq v$, is associated. We may write $a = (u, v)$ whenever it is not ambiguous. The node u is called the **source** and v is called the **target** of a.

Let $\emptyset \neq S \subset V$. We denote by $E_S := \delta(S)$ the set of edges in E with one endnode in S and one endnode in $V \backslash S$. E_S is called a **cut**. Similarly for directed graphs, $A_S := \delta(S)$ denotes the cut defined by S, where $A_S^+ \subseteq A_S$ is the set of edges with source in S and target in $V \backslash S$ and $A_S^- := A_S \backslash A_S^+$.

A graph is said to be **connected** if every cut is nonempty. A directed graph is **strongly connected** if both A_S^+ and A_S^- are nonempty for all $\emptyset \neq S \subset V$.

The set $E[S] \subset E$ (or $A[S] \subset A$) is the set of edges (arcs) with both endnodes in S. The corresponding subgraphs **induced** by S are defined as $G[S] := (S, E[S])$ or $G[S] := (S, A[S])$.

1.3.3 Polyhedra

A **polyhedron** P is defined as the intersection of finitely many affine halfspaces:

$$P = \{ f \in \mathbb{R}^n : \ Af \leq b \}.$$

In this thesis, every polyhedron is assumed to be rational, i. e., data is always given rational, so A is an $m \times n$ rational matrix and $b \in \mathbb{Q}^m$. We call the problem of optimising a linear function over P a **linear program** (LP). Let M and N denote two finite index sets. A **mixed integer set** X is given by

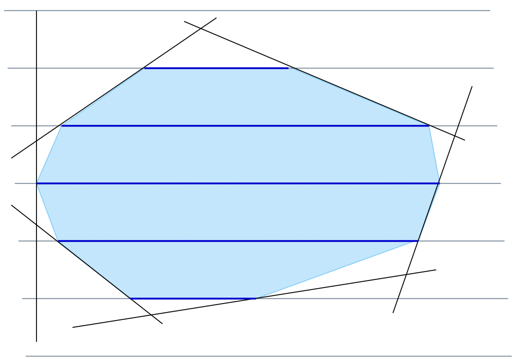


Figure 1.1: Mixed integer set and its convex hull

$$X = \{ (f, x) \in \mathbb{R}_+^M \times \mathbb{Z}_+^N : \ Af + Bx \leq b \}.$$

The problem of optimising a linear function over X, or equivalently over the convex hull $\mathrm{conv}(X)$, is called a **mixed integer program** (MIP). Note that (assuming rational data), $P_{MIP} := \mathrm{conv}(X)$ is a polyhedron (see Nemhauser & Wolsey [1988] for a proof).

Generally we do not know a set of linear inequalities defining P_{MIP}. Moreover, a MIP is $\mathcal{NP}$-hard in general. We cannot expect to derive a complete linear description of $\mathcal{NP}$-hard problems unless $\mathcal{NP} = \text{co-}\mathcal{NP}$, which follows from the equivalence of separation and optimisation (Grötschel et al. [1988]). For an introduction to complexity theory see for instance Schrijver [2003]. Nevertheless, it is usually possible to identify at least some classes of (strong) valid inequalities for P_{MIP}. From a theoretical as well as a computational point of view it is crucial to know whether they induce high dimensional faces or even define facets. An inequality

$$\gamma^T f \leq \pi \tag{1.1}$$

is said to be **valid** for a polyhedron P if $\gamma^T f \leq \pi$ for all $f \in P$. We call the set F of points in P that satisfy (1.1) with equality a **face** of P **induced** by (1.1):

$$F = \{\, f \in P :\ \gamma^T f = \pi \,\}.$$

A **facet** of P is an inclusion-wise maximal face F with $F \neq P$. Inequalities determining facets are called **facet-defining**. Any facet F satisfies $\dim(F) = \dim(P) - 1$.

Consider two inequalities, $\alpha^T f \leq \pi$ and $\gamma^T f \leq \pi$, which are valid for P, both of them having the same right hand side. Assume furthermore that all points in P are non-negative, that is $P = P \cap \mathbb{R}^n_+$. We say that the inequality defined by α is **at least as strong** as the one defined by γ, if $\gamma \leq \alpha$. If this is the case and there additionally exists an index i with $\gamma_i < \alpha_i$ then $\alpha^T f \leq \pi$ **dominates** $\gamma^T f \leq \pi$ or is said to be **stronger**.

The use of PORTA In most of the examples stated in this thesis we were interested in the dimension of faces induced by certain valid inequalities. Very often, previously developed theoretical results (facet theorems) could be applied, but sometimes we used the software package PORTA (Christof & Löbel [2005]) to calculate or estimate dimensions of faces. Consider the inequality

$$\gamma^T f + \beta^T x \leq \pi \tag{1.2}$$

valid for P_{MIP}. Generally PORTA is not able to directly compute the dimension $\dim_{MIP}$ of the face induced by (1.2). But we can do the following: Consider the LP-Relaxation of P_{MIP} after adding (1.2) to the initial formulation:

$$P_{LP} = \mathrm{conv}\{\, (f, x) \in \mathbb{R}^M_+ \times \mathbb{R}^N_+ :\ Af + Bx \leq b, \quad \gamma^T f + \beta^T x \leq \pi \,\}$$

and the bounded integer polytope:

$$P_{IP} = \mathrm{conv}\{\, (f, x) \in \mathbb{Z}^M_+ \times \mathbb{Z}^N_+ :\ Af + Bx \leq b, \quad \gamma^T f + \beta^T x \leq \pi, \quad f \leq u_1, x \leq u_2 \,\}$$

with $(u_1, u_2) \in \mathbb{Z}^M_+ \times \mathbb{Z}^N_+$ chosen appropriately. It is obvious that

$$P_{IP} \subseteq P_{MIP} \subseteq P_{LP}$$

and that (1.2) is valid for all three polyhedra. Additionally assume

$$\dim(P_{IP}) = \dim(P_{MIP}) = \dim(P_{LP}).$$

The dimensions of P_{IP} and P_{LP} as well as the dimensions $\dim_{LP}$, $\dim_{IP}$ of the faces induced by (1.2) can be evaluated with PORTA (by using the commands *dim*, *traf*, *vint* and *fctp*) at least for small instances and

$$\dim_{IP} \leq \dim_{MIP} \leq \dim_{LP} \tag{1.3}$$

holds, which can be used to estimate $\dim_{MIP}$. Moreover, if (1.2) defines a facet of P_{IP} then it defines a facet of P_{MIP} and if (1.2) does not define a facet of P_{LP} then it does not define a facet of P_{MIP}. (This holds only if $\dim(P_{IP}) = \dim(P_{LP})$ can be ensured.) These observations have been used whenever referring to the dimension of a face of a polyhedron of the form P_{MIP} and no theoretical result could be applied.

Note that P_{IP} has to be bounded because PORTA needs to enumerate all integer points in P_{IP} to calculate its dimension.

Chapter 2

Network Design Problems

2.1 Introduction

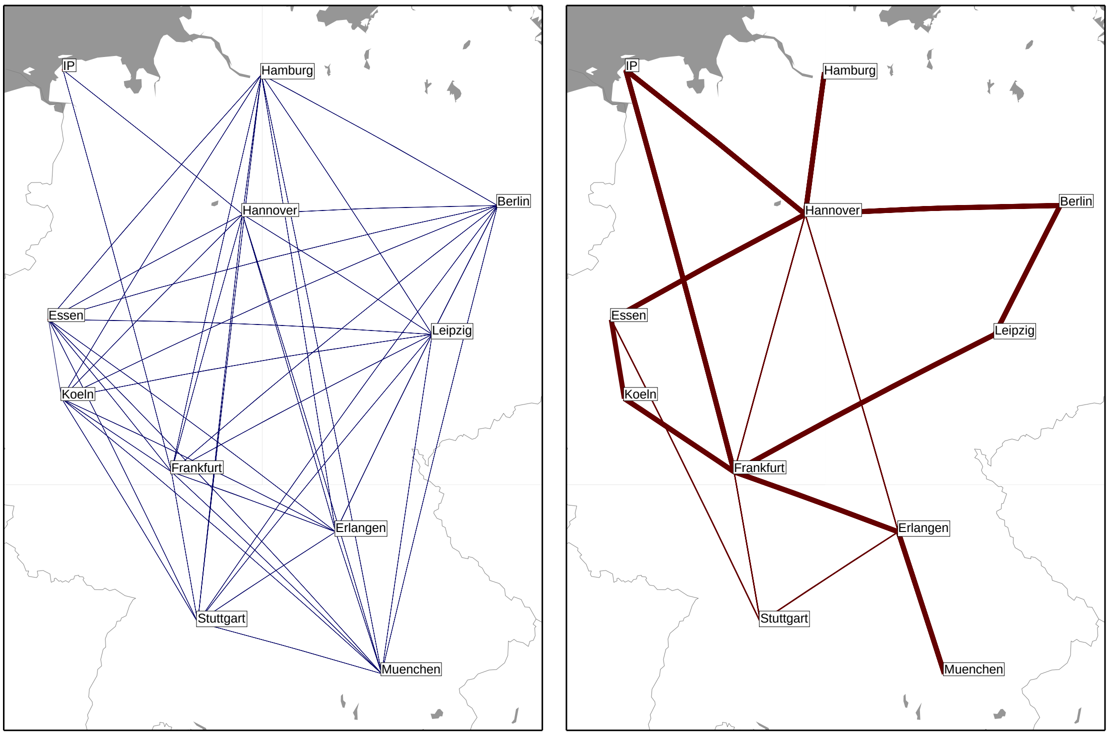
In this thesis we consider routing and capacity installation in the design of telecommunication networks. The presented formulations serve as relaxations and occur as sub-problems of larger and more complex problems that may include additional constraints and requirements such as survivability, hop limits, costs for hardware at the nodes of the network and more.

Literature Review and Problem Description Depending on the practical background there are many ways to define a telecommunication network design problem. We will briefly classify the most important models and will state the corresponding references.

Given a telecommunication network, communication demand of multiple commodities between certain locations has to be routed over the network. A routing (or flow) satisfying all given demands has to be assigned to the network and capacity has to be installed that suffices to accommodate the data flow. A network design problem or network loading problem consists of finding such a capacity and flow assignment that minimises the overall installation cost.

This problem has been studied in many variants with respect to network layout, capacity usage and the way of routing. The routing can be done by sending all flow on a single path between the endnodes of a point-to-point demand (*non-bifurcated* or *single-path* routing) or by considering several paths for every commodity (*bifurcated* routing). Single-path routing has been investigated by Brockmüller et al. [1998] and Hoesel et al. [2004]. The polyhedral study of models with bifurcated routing as considered in this thesis was initiated in a series of articles by Magnanti & Mirchandani [1993] and Magnanti et al. [1993, 1995]. Important extensions are from Bienstock & Günlük [1996].

In most of the practical applications the available technologies have a discrete structure, where capacity is restricted to a finite set of values. This might be modelled in different ways. For every link one may select exactly one capacity from a finite set of possible facilities (*explicit link capacities*) or alternatively every base capacity can be installed several times on every link up to a potentially given upper bound (*modular link capacities*). The base capacities might additionally be divisible. Dahl & Stoer [1994, 1998] consider explicit link capacities. Fundamental work for models with divisible modular link capacities up to three technologies was done by Magnanti & Mirchandani [1993], Magnanti et al. [1993, 1995], Bienstock & Günlük [1996] and Chopra et al. [1998]. Explicite as well as divisible modular link capacities were studied by Wessäly [2000]. In this thesis we investigate



(i) Supply graph

(ii) A capacity assignment: At most one of two technologies is selected for every link to satisfy all demands.

Figure 2.1: *G-WiN – German Research Network* [2005] — The data is taken from *SNDlib 1.0 – Survivable Network Design Data Library* [2005].

very general problems with modular link capacities that are not necessarily divisible as it was done for instance by Atamtürk [2002]. To model explicit link capacities our problems simply have to be extended by additional constraints.

The literature discerns three possible ways of capacity usage. A link might be directed, offering its capacity for flow in one direction only (*DIrected* capacity usage). If a link is undirected, the installed undirected capacity may either be shared between the two possible flow directions (*UNdirected* capacity usage) or it can be consumed by each of the two flow directions independently (*BIdirected* capacity usage). For DIrected capacity assignment it is referred to Bienstock et al. [1995], Chopra et al. [1998] and Atamtürk [2002]. BIdirected problems were mainly studied by Bienstock & Günlük [1996] and Günlük [1999] whereas a detailed analysis of UNdirected models can be found in Baharona [1994], Magnanti & Mirchandani [1993] and Magnanti et al. [1993, 1995]. We will consider all three forms of capacity usage in this thesis.

In the last decade it has become more and more important to consider *survivability* of telecommunication networks. In addition to the mentioned concepts of network design one wants to protect networks against certain failure situations such as cable cuts or hardware failures. We do not consider survivability here. For some basic results on survivability of telecommunication networks the reader is referred to Dahl & Stoer [1994, 1998], Alevras et al. [1996], Magnanti & Wang [1997], Balakrishnan et al. [1998] and Bienstock & Muratore [2000]. A good review can be found in Wessäly [2000], whereas Rajan & Atamtürk [2002*a*,*c*, 2004] present some recent work and new ideas.

Outline of this chapter This chapter presents the models investigated in this thesis. We discuss the given parameters and introduce variables as well as inequalities. Finally, we define the polyhedra corresponding to the considered network design problems.

2.2 Mathematical models

2.2.1 Parameters

Underlying graphs The **telecommunication network** or **supply graph** can be directed or undirected depending on the problem type. We denote it by $G = (V, A)$ for DIrected problems or $G = (V, E)$ for BIdirected and UNdirected problems. Nodes can be interpreted as being cities or locations whereas the arcs (or edges) represent connections or **links** between these locations (for instance by cable or some kind of radio contact). A supply graph is supposed to be connected and not to contain loops, but we allow parallel arcs and edges.

A provider of a telecommunication network has to face customer demands between some of the locations V.

Demands and Commodities A **demand** is an arc $a = (u, v)$ of the digraph $H = (V, D)$ (not necessarily connected), which we call **demand graph**. The demand graph is assumed to be simple, i. e., there are neither loops nor parallel arcs. A demand value $t_a \in \mathbb{Z}_+ \setminus \{0\}$ is assigned to every given demand $a = (u, v) \in D$. We have to establish a flow in G of t_a from u to v.

With the demands we associate a finite set of **commodities** K. For every commodity $k \in K$ there exists a function d^k that assigns a non-negative integer to every node of the supply graph $d^k : V \to \mathbb{Z}_+, i \mapsto d^k_i$

We call d^k_i the **net demand** of commodity k at node i. In the literature on multi-commodity network flow problems there are mainly two approaches related to the definition of commodities and net demands. The first is to consider one commodity for every demand, resulting in $|K| \in O(|V|^2)$ commodities ($K = D$). For every node $i \in V$ and every commodity $k = (u, v) \in D$ we can define:

$$d^k_i := \begin{cases} -t_{(u,v)} & i = v \\ t_{(u,v)} & i = u \\ 0 & \text{else} \end{cases} \quad i \in V, k = (u, v) \in K$$

This concept is called **disaggregated demands**.

The second approach is that of **aggregated demands**. One defines a commodity for each node that is source of at least one demand. Here $|K| \in O(|V|)$. The net demand of commodity $k = u \in K \subseteq V$ at node i is therefore defined the following way:

$$d^k_i := \begin{cases} -t_{(u,i)} & u \neq i, (u, i) \in D \\ \sum_{(u,v)\in D} t_{uv} & u = i \\ 0 & \text{else} \end{cases} \qquad i \in V, k = u \in K$$

Aggregating demands leads to a significantly reduced problem size since a feasible routing has to be found for every commodity, respectively. There are $O(|A||V|)$ flow variables in the aggregated formulation opposed to $O(|A||V|^2)$ flow variables in the disaggregated formulation.

A drawback of the aggregated formulation is that it often makes it impossible to formulate additional (demand-dependent) constraints, as for instance hop limits.

If not stated otherwise we will refer to a commodity simply as a function $d^k : V \to \mathbb{Z}_+, i \mapsto d_i^k$ with the property that

$$\sum_{i \in V} d_i^k = 0. \tag{2.1}$$

To satisfy the given demands, they have to be routed over the network. Capacity is provided on the links of the network in order to accommodate a feasible routing.

Capacities A finite set of installable **technologies** T is given. The literature often refers to the set T as **link designs** or **facilities**. We will use all three synonyms. Every technology $t \in T$ has a **base capacity** $c^t \in \mathbb{Z}_+ \backslash \{0\}$, which for instance reflects a certain bandwidth. Each of the facilities can be installed several times on every link of the network.

Note that in this thesis arc- or edge-dependent sets of installable technologies are not considered. However, all of the strong valid inequalities for network design polyhedra and most of the results are extendable to this more general case.

2.2.2 Variables

Link design variables There is an integer variable for every technology and every link, which we denote by $x_a^t \in \mathbb{Z}_+$ or $x_e^t \in \mathbb{Z}_+$. This variable indicates how many times facility t is installed on a given arc $a \in A$ or edge $e \in E$. Hence

$$\sum_{t \in T} c^t x_a^t, \qquad \sum_{t \in T} c^t x_e^t$$

is the total capacity available on a (or e). Given a subset $A_1 \subseteq A$ or $E_1 \subseteq E$ and a facility $t \in T$, we define

$$x^t(A_1) := \sum_{a \in A_1} x_a^t, \qquad x^t(E_1) := \sum_{e \in E_1} x_e^t.$$

Thus $c^t x^t(A_1)$ gives the total capacity that is installed on the set of arcs A_1 with respect to the technology t.

Flow variables In the following let Q be a subset of the commodities K and $k \in K$. Flow is always directed, independent of the type of the supply graph. Given an arc $a = (u, v) \in A$, we denote by $f_a^k \in \mathbb{R}_+$ the flow on a (from u to v) with respect to k. The notation for sums will be shortened the following way:

$$f^Q(A_1) := \sum_{a \in A_1,\, k \in Q} f_a^k,$$

with $A_1 \subseteq A$. For simplicity we define $f^k(A_1) := f^{\{k\}}(A_1)$ and $f_a^Q := f^Q(\{a\})$.

With every undirected edge $e = ij$ and every commodity $k \in K$ two flow variables are associated. The flow from i to j is denoted by $f_{ij}^k \in \mathbb{R}_+$ and the one from j to i is denoted by $f_{ji}^k \in \mathbb{R}_+$. Considering the commodity subset Q we write $f_{ij}^Q := \sum_{k \in Q} f_{ij}^k$ and $f_{ji}^Q := \sum_{k \in Q} f_{ji}^k$. Given a subset E_1 of the network edges, shortening the notation for sums as for directed supply graphs would be ambiguous in general, because of the two possible flow directions for every edge in E_1. But if E_1 is a subset of a cut E_S with $S \subset V$, there are two canonical ways to orientate the edges in E_1. We denote by

$$f^Q(E_1^+) := \sum_{\substack{e=ij \\ i\in S,\, j\in V\setminus S}} f_{ij}^Q \quad \text{and} \quad f^Q(E_1^-) := \sum_{\substack{e=ij \\ i\in S,\, j\in V\setminus S}} f_{ji}^Q$$

the total flow from S to $V\setminus S$ and from $V\setminus S$ to S on $E_1 \subseteq E$ with respect to Q. We use $f^k(E_1^+)$ and $f^k(E_1^-)$ instead of $f^{\{k\}}(E_1^+)$ and $f^{\{k\}}(E_1^-)$.

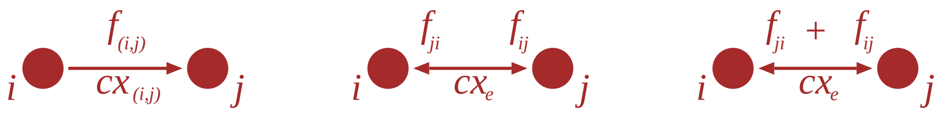
2.2.3 Inequalities

Flow conservation constraints The following inequalities ensure a feasible flow (routing) for every given commodity such that all demands are satisfied. We state them for directed and undirected supply graphs respectively.

$$\sum_{a\in\delta^+(i)} f_a^k - \sum_{a\in\delta^-(i)} f_a^k = d_i^k \quad k \in K, i \in V \tag{2.2}$$

$$\sum_{e=ij\in\delta(i)} f_{ij}^k - \sum_{e=ij\in\delta(i)} f_{ji}^k = d_i^k \quad k \in K, i \in V \tag{2.3}$$

Capacity Constraints We distinguish three problem types. Given a directed supply graph, the flow on every arc of the network is not allowed to exceed the installed capacity (**DIrected**). For undirected supply graphs the capacity available on an edge $e = ij$ may be used by the data flow of both directions ij and ji independently (**BIdirected**) or the capacity is shared between them (**UNdirected**).



(i) DIrected, Flow and capacity are directed.

(ii) BIdirected, Both directions may independently use the installed capacity.

(iii) UNdirected, Both directions share the installed capacity.

Figure 2.2: Capacity usage – single-facility, single-commodity

The three corresponding capacity constraints are as follows:

DIrected capacity constraints:

$$\sum_{k\in K} f_a^k \leq \sum_{t\in T} c^t x_a^t, \quad a \in A \tag{2.4}$$

BIdirected capacity constraints:

$$\begin{aligned} \sum_{k\in K} f_{ij}^k &\leq \sum_{t\in T} c^t x_e^t \quad e = ij \in E \\ \sum_{k\in K} f_{ji}^k &\leq \sum_{t\in T} c^t x_e^t, \quad e = ij \in E \end{aligned} \tag{2.5}$$

UNdirected capacity constraints:

$$\sum_{k\in K}(f_{ij}^k+f_{ji}^k) \;\le\; \sum_{t\in T} c^t x_e^t, \quad e=ij\in E \tag{2.6}$$

Nonnegativity Constraints

$$0 \;\le f_a^k, x_a^t \qquad a\in A, k\in K, t\in T \tag{2.7}$$
$$0 \;\le f_{ij}^k, f_{ji}^k, x_e^t \quad e=ij\in E, k\in K, t\in T \tag{2.8}$$

Bound constraints The number of base capacities that can be installed can be limited for every technology $t\in T$ and can even depend on the link of the network.

$$x_a^t \le u_a^t \quad a\in A, t\in T \tag{2.9}$$
$$x_e^t \le u_e^t \quad e\in E, t\in T \tag{2.10}$$

with u_a^t, $u_e^t \in \mathbb{Z}_+\backslash\{0\}$ for all $a\in A$, $e\in E$, $t\in T$.

2.2.4 The models

We will now define the multi-commodity, multi-facility **network design polyhedra** corresponding to the three problem types DIrected, BIdirected and UNdirected that are going to be investigated in this thesis. If no bound constraints are required we define

$$NDP^{DI} := \text{conv}\{\,(f,x)\in\mathbb{R}^{|K||A|}\times\mathbb{Z}^{|T||A|}:\ (f,x) \text{ satisfies (2.2), (2.4), (2.7)}\,\}$$
$$NDP^{BI} := \text{conv}\{\,(f,x)\in\mathbb{R}^{2|K||E|}\times\mathbb{Z}^{|T||E|}:\ (f,x) \text{ satisfies (2.3), (2.5), (2.8)}\,\}$$
$$NDP^{UN} := \text{conv}\{\,(f,x)\in\mathbb{R}^{2|K||E|}\times\mathbb{Z}^{|T||E|}:\ (f,x) \text{ satisfies (2.3), (2.6), (2.8)}\,\}$$

Note that these polyhedra depend on the underlying graphs, on the set of commodities and on the set of facilities. For simplicity of notation, we do not write them as a function of $G=(V,A)$, $G=(V,E)$, K or T. If some special properties of those parameters are required, we will explicitly state them.

Finally, let $u\in\mathbb{Z}_+\backslash\{0\}^{A\times T}$ (or $u\in\mathbb{Z}_+\backslash\{0\}^{E\times T}$) be the vector defining the bounds for the link design variables of every link and every facility. We write

$$NDP^{DI}(u) := \text{conv}\{\,(f,x)\in NDP^{DI}:\ (f,x) \text{ satisfies (2.9)}\,\}$$
$$NDP^{BI}(u) := \text{conv}\{\,(f,x)\in NDP^{BI}:\ (f,x) \text{ satisfies (2.10)}\,\}$$
$$NDP^{UN}(u) := \text{conv}\{\,(f,x)\in NDP^{UN}:\ (f,x) \text{ satisfies (2.10)}\,\}$$

These polyhedra model network design problems with bifurcated routing and modular link capacities. Adding generalised upper bound (GUB) constraints of the form

$$\sum_{t\in T} x_a^t \le 1 \;\; \forall a\in A, \qquad \sum_{t\in T} x_e^t \le 1 \;\; \forall e\in E$$

to our formulations yields problems with explicit link capacities. These inequalities model the common practical requirement that only one facility can be installed on a given arc or edge of the network and that it can be installed at most one time. We will not investigate such problems but in Chapter 7 the usefulness of the cutting planes developed in this thesis in the presence of GUB constraints will be tested. Note that GUB constraints imply bound constraints with $u_a^t = 1$ for all $a \in A, t \in T$ (or $u_e^t = 1$ for all $e \in E, t \in T$).

Objective function We want to minimise the overall cost of a capacity assignment that allows for a feasible routing and do not consider flow costs in this thesis. Let $\kappa_a^t \in \mathbb{Z}_+$ be the cost of installing the technology $t \in T$ on arc $a \in A$ and similar let $\kappa_e^t \in \mathbb{Z}_+$ be the cost of installing facility t on edge $e \in E$. Then the objective can be formalised as follows:

DIrected problems: $$\min \sum_{t \in T, a \in A} \kappa_a^t x_a^t$$

BIdirected and UNdirected problems: $$\min \sum_{t \in T, e \in E} \kappa_e^t x_e^t$$

The problem of minimising a linear function over a network design polyhedron will be called a **network design problem**. Note that the defined network design problems are $\mathcal{NP}$-hard already for very special cases, see for instances Bienstock & Günlük [1996], Chopra et al. [1998] and Atamtürk [2002].

2.3 Summary

We have started by classifying the type of models that will be considered in this thesis. We will study problems with bifurcated routing and a finite set of installable base capacities. We will consider bounded design variables as well as unbounded design variables. Three types of capacity usage will be distinguished: DIrected, BIdirected and UNdirected.

All given parameters have been discussed. We have introduced all the variables used and all necessary constraints. Finally we have defined the polyhedra corresponding to the different problem types.

Chapter 3

Mixed-Integer Rounding - *MIR*

3.1 Introduction

Literature review *Mixed-Integer Rounding* (*MIR*) is a very basic and general tool in mixed integer programming and has many applications. The idea itself is from Gomory [1960]. He introduced the so-called *Gomory mixed integer cut*, which can be seen as *MIR* with a valid base inequality taken from the simplex tableau (see for instance Marchand & Wolsey [1998]). Algorithms to separate *Gomory mixed integer cuts* are included in state-of-the-art MIP-solvers such as *CPLEX* (ILOG [2005]), *Xpress* (Dash Optimizations [2005]) or *SCIP* (Achterberg [2005]) and are crucial for the efficiency of such solvers (see Bixby et al. [2000]). The general *MIR*-cut for arbitrary base inequalities can be found for instance in Nemhauser & Wolsey [1988] and Wolsey [1998].

MIR was somehow rediscovered in the nineties of the last century. First of all, computational results showed that *Gomory mixed integer cuts* can be effective when implemented in a Branch & Cut framework, in contrast to Gomory's cutting plane algorithm of the early sixties (see Balas et al. [1996] and Bixby et al. [2000]). Moreover, it was observed that several general methods of generating inequalities are equivalent (specifically disjunctive, split, Gomory mixed and *MIR* inequalities) and that certain families of strong inequalities are in fact *MIR*-inequalities (see Marchand [1997] and Marchand & Wolsey [1998]). The latter is in particular true for most of the well-known cuts for network design problems (see Chapter 4, Chapter 5 and Chapter 6).

Outline of this chapter In this chapter we will present some common techniques for deriving strong valid inequalities from mixed integer sets using *MIR*. Those techniques will be part of a *MIR* procedure for network design problems, which will be discussed in Section 3.4.

We start with the general *MIR*-inequality for $\leq$- and $\geq$-base inequalities. We explain the relation to superadditivity and subadditivity, make some statements about scaling and show how to use *MIR* safely from a numerical point of view. Section 3.2 explains the use of *MIR* for lifting and in Section 3.3 we show how to exploit the special structure of mixed integer sets when bounds are given.

The basic idea To explain the basic idea of *Mixed Integer Rounding* it suffices to consider a two-variable set (Figure 3.1) and a valid inequality that is strengthened by a simple rounding step. The corresponding result is easily generalised to higher dimensions.

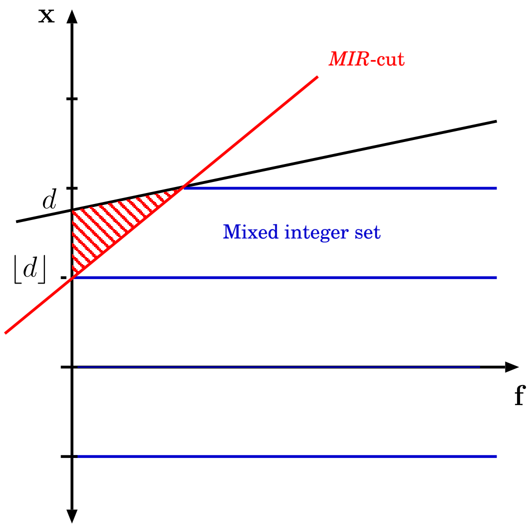


Figure 3.1: A *Mixed-Integer Rounding* cut

Lemma 3.1 *Consider* $(f, x) \in \mathbb{R}_+ \times \mathbb{Z}$ *satisfying the inequality*

$$af + x \leq d$$

with $a, d \in \mathbb{Q}$ *and* $a \leq 0$. (f, x) *also satisfies the following inequality:*

$$\frac{a}{1 - \langle d \rangle} f + x \leq \lfloor d \rfloor.$$

Proof. The result is trivial if $\langle d \rangle = 0$. Suppose $\langle d \rangle > 0$. If $a = 0$ the validity of $x \leq \lfloor d \rfloor$ follows from the integrality of x. Let a be negative.

If $x \leq \lfloor d \rfloor$ we have $x - \lfloor d \rfloor \leq 0 \implies (x - \lfloor d \rfloor)(1 - \langle d \rangle) \leq 0$ since $0 < \langle d \rangle < 1$. But then $(x - \lfloor d \rfloor)(1 - \langle d \rangle) \leq -af$ because $-af \geq 0$.

If $x \geq \lceil d \rceil = \lfloor d \rfloor + 1$ we have $-(x - \lfloor d \rfloor) \leq -1$. Rewriting the valid inequality $af + x \leq d$ results in $(x - \lfloor d \rfloor) \leq \langle d \rangle - af$. Combining those two inequalities with weights $\langle d \rangle$ and 1 respectively gives $(x - \lfloor d \rfloor)(1 - \langle d \rangle) \leq -af$. ∎

In the following the last result will be used to define the general *MIR*-inequality and to prove its validity. Given $(f, x) \in \mathbb{R}_+^M \times \mathbb{Z}_+^N$ consider the following base inequality:

$$\sum_{j \in M} a_j f_j + \sum_{j \in N} c_j x_j \leq d \tag{3.1}$$

where M denotes the (finite) set of continuous variables, N denotes the (finite) set of integer variables and a_j, c_j, d are rational numbers. Inequality (3.1) may arise as a linear combination of rows of a general mixed integer program. With (3.1) we associate two simple mixed integer sets:

$$X = \{ (f, x) \in \mathbb{R}_+^M \times \mathbb{Z}_+^N : (f, x) \text{ satisfies (3.1)} \}$$
$$\text{and} \quad Y = \{ (f, x) \in \mathbb{R}_+^M \times \mathbb{Z}_+^N : (f, x) \text{ satisfies (3.1)}, \quad x_j \leq u_j, j \in N \}$$

where $u_j \in \mathbb{Z}_+ \backslash \{0\} \ \forall j \in N$.

Theorem 3.2 *The following inequality is valid for X and Y and defines the* Mixed-Integer Rounding cut*:*

$$\sum_{j \in M} \frac{a_j^-}{1-\langle d\rangle} f_j \;+\; \sum_{j \in N} \Big(\lfloor c_j \rfloor + \frac{(\langle c_j\rangle - \langle d\rangle)^+}{1-\langle d\rangle} \Big) x_j \;\le\; \lfloor d \rfloor. \tag{3.2}$$

Proof. The inequality (3.1) can be relaxed by deleting flow variables if $a_j \ge 0$ and by rounding down coefficients c_j for integer variables if $\langle c_j\rangle \le \langle d\rangle$:

$$\sum_{a_j<0} a_j f_j + \sum_{\langle c_j\rangle \le \langle d\rangle} \lfloor c_j \rfloor x_j + \sum_{\langle c_j\rangle > \langle d\rangle} \lceil c_j \rceil x_j - \sum_{\langle c_j\rangle > \langle d\rangle} (1-\langle c_j\rangle) x_j \le d$$

Note that from $\langle c_j\rangle > \langle d\rangle$ follows $\langle c_j\rangle > 0$ and hence $c_j = \lceil c_j \rceil - 1 + \langle c_j\rangle$. Observing that

i) $-\sum_{\langle c_j\rangle > \langle d\rangle} (1-\langle c_j\rangle) x_j + \sum_{a_j<0} a_j f_j \qquad \le 0$

ii) $\sum_{\langle c_j\rangle \le \langle d\rangle} \lfloor c_j \rfloor x_j + \sum_{\langle c_j\rangle > \langle d\rangle} \lceil c_j \rceil x_j \qquad \in \mathbb{Z}$

we use Lemma 3.1 to obtain:

$$\sum_{j \in M} \frac{a_j^-}{1-\langle d\rangle} f_j + \sum_{\langle c_j\rangle \le \langle d\rangle} \lfloor c_j \rfloor x_j + \sum_{\langle c_j\rangle > \langle d\rangle} \lceil c_j \rceil x_j - \sum_{\langle c_j\rangle > \langle d\rangle} \frac{1-\langle c_j\rangle}{1-\langle d\rangle} x_j \le \lfloor d \rfloor.$$

Noting that

$$\lceil c_j \rceil - \frac{1-\langle c_j\rangle}{1-\langle d\rangle} = \lfloor c_j \rfloor + \frac{\langle c_j\rangle - \langle d\rangle}{1-\langle d\rangle} \qquad \text{if} \quad \langle c_j\rangle > 0$$

concludes the proof. ∎

The *MIR* inequality (3.2) often strengthens the base inequality (3.1). Since X and Y can be seen as relaxations of more complex mixed integer sets, (3.2) provides a very general cutting plane that can be used in Branch & Cut algorithms to solve mixed integer programs. How to derive good base inequalities is one of the major questions considered in this thesis.

Remark 3.3 *If d is integer and hence $\langle d\rangle = 0$ inequality* (3.2) *reduces to*

$$\sum_{j \in M} a_j^- f_j \;+\; \sum_{j \in N} c_j x_j \;\le\; d,$$

which relaxes the base inequality (3.1).

The *MIR*-inequality is well defined and valid if $M = \emptyset$. In this case (3.2) is obviously at least as strong as the so-called *Chvátal-Gomory cut* (Gomory [1958] and Chvàtal [1973]) for pure integer sets:

$$\sum_{j \in N} \lfloor c_j \rfloor x_j \;\le\; \lfloor d \rfloor$$

since $\lfloor c_j \rfloor + \frac{(\langle c_j\rangle - \langle d\rangle)^+}{1-\langle d\rangle} \ge \lfloor c_j \rfloor$.

MIR inequalities are of algebraic nature. They do not always have a geometrical interpretation since scaling the base inequality with some constant may lead to a different *MIR* inequality. Given a positive rational number k the inequality

$$\sum_{j\in M} \frac{ka_j^-}{1-\langle kd\rangle} f_j \;+\; \sum_{j\in N}\Big(\lfloor kc_j\rfloor + \frac{(\langle kc_j\rangle - \langle kd\rangle)^+}{1-\langle kd\rangle}\Big)x_j \;\le\; \lfloor kd\rfloor \tag{3.3}$$

is also valid for X, Y and called a k-*MIR* inequality (Cornuejols et al. [2003]).

Example 3.4 *The* MIR *inequality of* $3.4x_1 + 3.7x_2 \le 8.2$, *with* $x_1, x_2 \in \mathbb{Z}_+$ *is* $3.25x_1 + 3.625x_2 \le 8$ *dominating the Chvátal-Gomory cut* $3x_1 + 3x_2 \le 8$. *But even stronger is the* 3-MIR *inequality* $10x_1 + 11x_2 \le 24$.

Superadditivity and subadditivity The function mapping coefficients of integer variables in (3.1) onto coefficients in (3.2) has a nice property, which turns out to be crucial for the theory of mixed integer sets and strong valid inequalities.

Definition 3.5 *A function* $F : \mathbb{R} \to \mathbb{R}$ *is* superadditive *if*

$$F(a) + F(b) \le F(a+b)$$

for all $a, b \in \mathbb{R}$. *A function* $G : \mathbb{R} \to \mathbb{R}$ *is* subadditive *if*

$$G(a) + G(b) \ge G(a+b)$$

for all $a, b \in \mathbb{R}$. *We write* $\overline{F}(a) = \lim_{t\searrow 0}\frac{F(at)}{t}$ *and* $\overline{G}(a) = \lim_{t\searrow 0}\frac{G(at)}{t}$ *if the limes exist.*

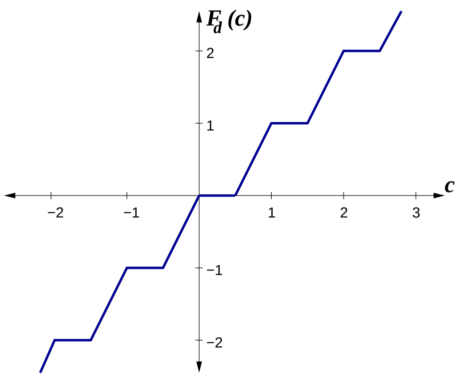


Figure 3.2: The superadditive *MIR* function F_d with $\langle d\rangle = 0.5$

It is well-known that the function

$$F_d : \mathbb{R} \to \mathbb{R}, \qquad F_d(c) = \lfloor c\rfloor + \frac{(\langle c\rangle - \langle d\rangle)^+}{1-\langle d\rangle},$$

with $d \in \mathbb{R}$, is superadditive and nondecreasing (see Nemhauser & Wolsey [1988] and Figure 3.2). Moreover $F_d(0) = 0$ and if $\langle d\rangle > 0$ then $\overline{F}_d(a) = \frac{a^-}{1-\langle d\rangle}$ $\forall a \in \mathbb{R}$. If otherwise $\langle d\rangle = 0$ then $\overline{F}_d(a) = a$ $\forall a \in \mathbb{R}$. Hence

$$\sum_{j\in M} \overline{F}_d(a_j) f_j + \sum_{j\in N} F_d(c_j)x_j \le F_d(d).$$

is the the *MIR* inequality (3.2) if $\langle d\rangle > 0$ and the base inequality (3.1) else.

Remark 3.6 *Note that $\sum_{j\in M}\overline{F}(a_j)f_j + \sum_{j\in N} F(c_j)x_j \le F(d)$ defines a valid inequality for X and Y whenever F is a superadditive nondecreasing function with $F(0) = 0$ and $\overline{F}(a_j)$ exists for all $j \in M$, which generalises Theorem 3.2 and is a crucial result in mixed integer programming (see Nemhauser & Wolsey [1988]).*

We state a corollary to Theorem 3.2 and a *MIR* inequality for $\geq$-base inequalities because this is the setting we will consider most of the time. Given the base inequality

$$\sum_{j\in M} a_j f_j \ + \ \sum_{j\in N} c_j x_j \ \geq \ d \tag{3.4}$$

define

$$\begin{aligned} X^{\geq} &= \{\, (f,x) \in \mathbb{R}^M_+ \times \mathbb{Z}^N_+ :\ (f,x) \text{ satisfies (3.4)} \,\} \\ \text{and} \quad Y^{\geq} &= \{\, (f,x) \in \mathbb{R}^M_+ \times \mathbb{Z}^N_+ :\ (f,x) \text{ satisfies (3.4)}, \quad x_j \le u_j, j \in N \,\}. \end{aligned}$$

where $u_j \in \mathbb{Z}_+ \backslash \{0\}\ \forall j \in N$.

Corollary 3.7 *The* MIR *inequality*

$$\sum_{j\in M} \frac{a_j^+}{r(d)} f_j + \sum_{j\in N} \big(\lceil c_j \rceil - \frac{(r(d) - r(c_j))^+}{r(d)}\big) x_j \quad \geq \quad \lceil d \rceil \tag{3.5}$$

is valid for $X^{\geq}$ and $Y^{\geq}$.

Proof. Multiplying (3.4) by -1, applying Theorem 3.2 and again multiplying by -1 results in

$$\sum_{j\in M} \frac{-(-a_j)^-}{1 - \langle -d \rangle} f_j + \sum_{j\in N} \big(- \lfloor -c_j \rfloor - \frac{(\langle -c_j \rangle - \langle -d \rangle)^+}{1 - \langle -d \rangle} \big) x_j \quad \geq \quad -\lfloor -d \rfloor.$$

Using that for all $\lambda \in \mathbb{R}$

i) $-(-\lambda)^- = \lambda^+$ ii) $-\lfloor -\lambda \rfloor = \lceil \lambda \rceil$ iii) $\langle -\lambda \rangle = 1 - r(\lambda)$

gives the desired result. ∎

Note that $G_d : \mathbb{R} \to \mathbb{R},\ G_d(c) = \lceil c \rceil - \frac{(r(d)-r(c))^+}{r(d)}$ is subadditive with $\overline{G}_d(a) = \frac{a^+}{r(d)}$ when $\langle d \rangle > 0$ and $\overline{G}_d(a) = a$ else. This follows from $G_d(c) = -F_{-d}(-c)$. Similar to the result above

$$\sum_{j\in M} \overline{G}_d(a_j) f_j + \sum_{j\in N} G_d(c_j) x_j \geq G_d(d).$$

gives the *MIR*-inequality (3.5) when $\langle d \rangle > 0$ and the base inequality (3.4) for $\langle d \rangle = 0$.

Numerics and integer coefficients A shortcoming of the *MIR* cut, compared to the Chvátal-Gomory cut, is that the rational coefficients $\overline{G}_d(a_j)$, $\overline{F}_d(a_j)$ and $F_d(c_j)$, $G_d(c_j)$ might be fractional with large denominators. Hence scaling *MIR* inequalities to obtain integer coefficients may cause numerical problems. This is known especially for implementations of *Gomory mixed integer cuts*. However, when solely considering base inequalities with integer coefficients and right hand side we can avoid such problems. Given $c \in \mathbb{Z}_+ \setminus \{0\}$, the corresponding $\frac{1}{c}$-*MIR* inequality can be scaled in such a way that all coefficients (and right hand side) are integers bounded by the coefficients (right hand side) of the base inequality. In the following we will simply scale the $\frac{1}{c}$-*MIR* inequality with the factor $r(d,c)$. For $a, c, d \in \mathbb{R}$ and $c > 0$ define

$$\mathcal{G}_{d,c}(a) := r(d,c) G_{\frac{d}{c}}(\tfrac{a}{c}) \quad \text{and} \quad \overline{\mathcal{G}}_{d,c}(a) := r(d,c)\overline{G}_{\frac{d}{c}}(\tfrac{a}{c})$$

Corollary 3.8 *Let $c \in \mathbb{Q}, c > 0$. The following $\frac{1}{c}$-*MIR *inequality is valid for $X^{\geq}$ and $Y^{\geq}$:*

$$\begin{aligned} &\sum_{j \in M} \overline{\mathcal{G}}_{d,c}(a_j) f_j + \sum_{j\in N} \mathcal{G}_{d,c}(c_j) x_j && \geq \mathcal{G}_{d,c}(d) \qquad (3.6)\\ \iff \quad &\sum_{j \in M} \overline{\mathcal{G}}_{d,c}(a_j) f_j + \sum_{j\in N} \big(r(d,c)\lceil \tfrac{c_j}{c}\rceil - (r(d,c) - r(c_j,c))^+\big) x_j && \geq r(d,c)\lceil \tfrac{d}{c} \rceil. \end{aligned}$$

If $a, d, c \in \mathbb{Z}, c > 0$ then both $\overline{\mathcal{G}}_{d,c}(a) \in \mathbb{Z}$ and $\mathcal{G}_{d,c}(a) \in \mathbb{Z}$. Moreover, the MIR *coefficients are bounded by the base coefficients:*

$$0 \leq |\overline{\mathcal{G}}_{d,c}(a)| \leq |a| \quad \textit{and} \quad 0 \leq |\mathcal{G}_{d,c}(a)| \leq |a|.$$

Proof. We divide (3.4) by c, apply Corollary 3.7 and arrive at

$$\sum_{j\in M} \overline{G}_{\frac{d}{c}}(\tfrac{a_j}{c}) f_j + \sum_{j \in N} \Big(\lceil \tfrac{c_j}{c} \rceil - \frac{(r(\frac{d}{c}) - r(\frac{c_j}{c}))^+}{r(\frac{d}{c})}\Big) x_j \quad \geq \quad \lceil \tfrac{d}{c} \rceil.$$

Multiplying with $c \cdot r(\frac{d}{c}) = r(d,c)$ gives (3.6). Suppose $a, d, c \in \mathbb{Z}, c > 0$. Then obviously $\overline{\mathcal{G}}_{d,c}(a) = a^+ \in \mathbb{Z}$ or $\overline{\mathcal{G}}_{d,c}(a) = a \in \mathbb{Z}$ and $\mathcal{G}_{d,c}(a) = \big(r(d,c)\lceil \frac{a}{c} \rceil - (r(d,c) - r(a,c))^+\big) \in \mathbb{Z}$.

Moreover $0 \leq |\overline{\mathcal{G}}_{d,c}(a)| \leq |a|$. It remains to show that $|\mathcal{G}_{d,c}(a)| \leq |a|$. It is easily checked that $\mathcal{G}_{d,c}(0) = 0$. First assume $a > 0$. Let $r(d,c) \leq r(a,c)$. With Lemma 3.11 follows

$$0 \leq \mathcal{G}_{d,c}(a) = r(d,c)\lceil \tfrac{a}{c} \rceil \leq r(a,c)\lceil \tfrac{a}{c}\rceil = a - (\lceil \tfrac{a}{c} \rceil - 1)(a - r(a,c)) \leq a$$

since $\lceil \frac{a}{c} \rceil \geq 1$ and $0 \leq r(a,c) \leq a$. Now let $r(d,c) > r(a,c)$. Then $r(a,c) < c \implies \langle \frac{a}{c} \rangle > 0$ and

$$\begin{aligned} 0 \leq \mathcal{G}_{d,c}(a) = r(d,c)\lceil \tfrac{a}{c}\rceil - (r(d,c) - r(a,c)) &= r(d,c)\lfloor \tfrac{a}{c} \rfloor + r(a,c) \\ &\leq c\lfloor \tfrac{a}{c} \rfloor + r(a,c) \\ &= c\lfloor \tfrac{a}{c} \rfloor + c\langle \tfrac{a}{c} \rangle = a. \end{aligned}$$

If $a < 0$ we use that $\mathcal{G}_{d,c}(a) = \mathcal{G}_{-d,c}(-a) + a$ (Lemma 3.11 iii)). From $0 \leq \mathcal{G}_{-d,c}(-a) \leq -a$ follows then $a \leq \mathcal{G}_{d,c}(a) \leq 0$. ∎

Similarly, we define

$$\overline{\mathcal{F}}_{d,c}(a) := r(-d,c)\overline{F}_{\frac{d}{c}}(\tfrac{a}{c}) \qquad \text{and}$$

$$\begin{aligned}\mathcal{F}_{d,c}(a) := r(-d,c)F_{\frac{d}{c}}(\tfrac{a}{c}) \qquad &= -r(-d,c)G_{\frac{-d}{c}}(\tfrac{-a}{c}) = -\mathcal{G}_{-d,c}(-a) \\ &= -(r(-d,c)\lceil\tfrac{-a}{c}\rceil - \big(r(-d,c) - r(-a,c)\big)^+) \\ &= r(-d,c)\lfloor\tfrac{a}{c}\rfloor + \big(r(-d,c) - r(-a,c)\big)^+.\end{aligned} \tag{3.7}$$

Note that $\overline{\mathcal{F}}_{d,c}(a) = -\overline{\mathcal{G}}_{-d,c}(-a)$. So, $\overline{\mathcal{F}}_{d,c}(a) = a^-$ if $\langle\frac{d}{c}\rangle > 0$ and $\overline{\mathcal{F}}_{d,c}(a) = a$ else. We can formulate the following valid inequality for X, Y:

$$\sum_{j\in M}\overline{\mathcal{F}}_{d,c}(a_j)f_j + \sum_{j\in N}\mathcal{F}_{d,c}(c_j)x_j \leq \mathcal{F}_{d,c}(d). \tag{3.8}$$

Again if $a, c, d \in \mathbb{Z}, c > 0$ then $0 \leq \overline{\mathcal{F}}_{d,c}(a), \mathcal{F}_{d,c}(a) \in \mathbb{Z}$ and $|\mathcal{F}_{d,c}(a)|, |\overline{\mathcal{F}}_{d,c}(a)| \leq |a|$. The formulas (3.6) and (3.8) should be used when implementing $\frac{1}{c}$-*MIR* inequalities. When separating such *MIR* inequalities within a Branch & Cut algorithm we are now sure not to worsen the condition of the underlying matrix, which is crucial for the correctness and effectiveness of those algorithms.

Example 3.9 *The following investigation have been made by using the software package PORTA (Christof & Löbel [2005]). Consider the integer knapsack set*

$$Y^{\geq}(u) = \{\, x \in \mathbb{Z}^4_+ : \; 4x_1 + 7x_2 + x_3 + 2x_4 \geq 13, \quad x_i \leq u, i \in \{1,..,4\}\,\}$$

where $u \in \mathbb{Z}_+ \cup \{\infty\}$. *Using formula* (3.6) *we calculate three possible* $\frac{1}{c}$-MIR *inequalities with* $c \in \{4, 7, 2\}$ *and state the dimension of the induced faces corresponding to* $\operatorname{conv}(Y^{\geq}(2))$ *and* $\operatorname{conv}(Y^{\geq}(\infty))$*:*

$\frac{1}{4} -$ MIR : $x_1 + 2x_2 + x_3 + x_4 \geq 4$ *dimension for* $Y^{\geq}(2) : 2$ *dimension for* $Y^{\geq}(\infty) : 3$
$\frac{1}{7} -$ MIR : $4x_1 + 6x_2 + x_3 + 2x_4 \geq 12$ *dimension for* $Y^{\geq}(2) : 3$ *dimension for* $Y^{\geq}(\infty) : 3$
$\frac{1}{2} -$ MIR : $2x_1 + 4x_2 + x_3 + x_4 \geq 7$ *dimension for* $Y^{\geq}(2) : 1$ *dimension for* $Y^{\geq}(\infty) : 3$

Both $\operatorname{conv} Y^{\geq}(2)$ *and* $\operatorname{conv} Y^{\geq}(\infty)$ *are full-dimensional. Note that the coefficients of all* MIR-*inequalities are not greater than the corresponding coefficients of the base inequality. For the polyhedron* $\operatorname{conv} Y^{\geq}(\infty)$ *all three* MIR-*inequalities are facet-defining. This is not the case for* $\operatorname{conv} Y^{\geq}(2)$. *In Section 3.3 techniques are presented that exploit the special structure of sets with bounded variables.*

A special case of Corollary 3.8 and inequality (3.6) is used frequently in the literature (Atamtürk [2002] and Bienstock & Günlük [1996] and Magnanti & Mirchandani [1993] and Chopra et al. [1998] and others):

Corollary 3.10 *If*

$$f \pm cx \geq d \tag{3.9}$$

is a valid base inequality for $(f, x) \in \mathbb{R}_+ \times \mathbb{Z}$ *with* $c \in \mathbb{R}, c > 0$ *then*

$$f \pm r(d,c)x \geq r(d,c)\lceil\frac{d}{c}\rceil \tag{3.10}$$

is also valid.

Proof. Set $\tilde{x} := -x^- \implies x = x^+ - \tilde{x}$ with $x^+, \tilde{x} \in \mathbb{Z}_+$. Rewrite (3.9):

$$f \pm c \cdot (x^+ - \tilde{x}) \geq d$$

and apply Corollary 3.8 using $r(d,c) \leq r(c,c) = r(-c,c) = c$ and $\lceil \frac{c}{c} \rceil = 1 = -\lceil \frac{-c}{c} \rceil$. ∎

We conclude with a lemma that will be needed several times and is based on the introduced notation.

Lemma 3.11 *Let $x \in \mathbb{R}$, $y \in \mathbb{R}_+$ $d, c, c_1 \in \mathbb{Z}_+$ and $z \in \mathbb{Z}$:*

i) $c\lceil \frac{d}{c} \rceil = d + c - r(d,c)$ *and* $r(d,c)\lceil \frac{d}{c} \rceil = d - (\lceil \frac{d}{c} \rceil - 1)(c - r(d,c))$

ii) If $\langle \frac{x}{y} \rangle > 0$ *then* $\langle \frac{x}{y} \rangle = 1 - \langle \frac{-x}{y} \rangle$ *and* $r(x,y) = y - r(-x,y)$

iii) If $\langle \frac{d}{c} \rangle > 0$ *then* $\mathcal{F}_{d,c}(-z) = \mathcal{F}_{-d,c}(z) - z$ *and* $\mathcal{G}_{d,c}(-z) = \mathcal{G}_{-d,c}(z) - z$

iv) If $c > d$ *and* $c \geq c_1$ *then*

$$\begin{aligned}
\mathcal{F}_{d,c}(c_1) &= (c_1 - d)^+, & \mathcal{G}_{d,c}(c_1) &= \min(d, c_1) \\
\mathcal{F}_{-d,c}(c_1) &= (d - c + c_1)^+, & \mathcal{G}_{-d,c}(c_1) &= \min(c - d, c_1) \\
\mathcal{F}_{d,c}(-c_1) &= -\min(c - d, c_1), & \mathcal{G}_{d,c}(-c_1) &= -(d - c + c_1)^+ \\
\mathcal{F}_{-d,c}(-c_1) &= -\min(d, c_1), & \mathcal{G}_{-d,c}(-c_1) &= -(c_1 - d)^+.
\end{aligned}$$

Proof. i) If $\langle \frac{d}{c} \rangle = 0 \iff r(d,c) = c$, then $c\lceil \frac{d}{c} \rceil = r(d,c)\lceil \frac{d}{c} \rceil = d$. Else if $\langle \frac{d}{c} \rangle > 0 \implies c\lceil \frac{d}{c} \rceil = c(\frac{d}{c} + 1 - \langle \frac{d}{c} \rangle) = d + c - c\langle \frac{d}{c} \rangle = d + c - r(d,c)$. It follows that $r(d,c)\lceil \frac{d}{c} \rceil = d + c - r(d,c) - (c - r(d,c))\lceil \frac{d}{c} \rceil = d - (\lceil \frac{d}{c} \rceil - 1)(c - r(d,c))$.

ii) $\langle \frac{-x}{y} \rangle = \frac{-x}{y} - \lfloor \frac{-x}{y} \rfloor = -\frac{x}{y} + \lceil \frac{x}{y} \rceil = -\frac{x}{y} + \lfloor \frac{x}{y} \rfloor + 1 = 1 - \langle \frac{x}{y} \rangle$.
$r(x,y) = \langle \frac{x}{y} \rangle y = (1 - \langle \frac{-x}{y} \rangle)y = y - r(-x,y)$

iii) $\langle \frac{d}{c} \rangle > 0$ then $r(d,c) < c$ and $r(-d,c) = c - r(d,c)$. We can write

$$\begin{aligned}
\mathcal{F}_{d,c}(-z) &= (c - r(d,c))\lfloor \tfrac{-z}{c} \rfloor + \big(c - r(d,c) - r(z,c)\big)^+ \\
&= (r(d,c) - c)\lceil \tfrac{z}{c} \rceil + \big(c - r(d,c) - r(z,c)\big)^+
\end{aligned}$$

and

$$\mathcal{F}_{-d,c}(z) - z = r(d,c)\lfloor \tfrac{z}{c} \rfloor + \big(r(d,c) - r(-z,c)\big)^+ - z.$$

(a) $\langle \frac{z}{c} \rangle = 0 \implies \mathcal{F}_{d,c}(-z) = (r(d,c) - c)\frac{z}{c} + \big(c - r(d,c) - c\big)^+ = r(d,c)\frac{z}{c} - z = \mathcal{F}_{-d,c}(z) - z$.

(b) Assume that $\langle \frac{z}{c} \rangle > 0$ and $c - r(d,c) - r(z,c) > 0$. Hence $\mathcal{F}_{d,c}(-z) = (r(d,c) - c)\lceil \frac{z}{c} \rceil + \big(c - r(d,c) - r(z,c)\big) = (r(d,c) - c)\lfloor \frac{z}{c} \rfloor - r(z,c) = r(d,c)\lfloor \frac{z}{c} \rfloor - z = \mathcal{F}_{-d,c}(z) - z$.

(c) Finally suppose $\langle \frac{z}{c} \rangle > 0$ and $c - r(d,c) - r(z,c) \leq 0$. Thus $\mathcal{F}_{d,c}(-z) = (r(d,c) - c)\lceil \frac{z}{c} \rceil = r(d,c)\lfloor \frac{z}{c} \rfloor + r(d,c) - c\lceil \frac{z}{c} \rceil = r(d,c)\lfloor \frac{z}{c} \rfloor + r(d,c) - z - c + r(z,c) = r(d,c)\lfloor \frac{z}{c} \rfloor + r(d,c) - z - r(-z,c) = \mathcal{F}_{-d,c}(z) - z$.

Using equation (3.7) gives $\mathcal{G}_{d,c}(-a) = -\mathcal{F}_{-d,c}(a) = -\mathcal{F}_{d,c}(-a) - a = \mathcal{G}_{-d,c}(a) - a$.

iv) We prove that $\mathcal{F}_{d,c}(c_1) = (c_1 - d)^+$ and $\mathcal{F}_{-d,c}(c_1) = (d - c + c_1)^+$. The rest follows from Lemma 3.11 iii) and Equation (3.7).

   (a) $\mathcal{F}_{d,c}(c_1) = \lfloor \frac{c_1}{c} \rfloor r(-d,c) + \big(r(-d,c) - r(-c_1,c)\big)^+$. If $c > c_1$ then $\mathcal{F}_{d,c}(c_1) = (c - d - c + c_1)^+ = (c_1 - d)^+$. Else if $c = c_1$ then $\mathcal{F}_{d,c}(c_1) = (c-d) + (c-d-c)^+ = (c-d) = (c_1 - d)^+$.

   (b) $\mathcal{F}_{-d,c}(c_1) = \lfloor \frac{c_1}{c} \rfloor r(d,c) + \big(r(d,c) - r(-c_1,c)\big)^+$. If $c > c_1$ then $\mathcal{F}_{-d,c}(c_1) = (d - c + c_1)^+$. Else if $c = c_1$ then $\mathcal{F}_{-d,c}(c_1) = d + (d-c)^+ = d = (d - c + c_1)^+$. ∎

Summary In this section we have introduced *MIR* and made some statements about superadditivity and numerics. The inequalities (3.6) and (3.8) will be used when deriving strong valid inequalities throughout the rest of this thesis.

3.2 *MIR*, Superadditivity and Lifting

We cannot give a complete introduction to the concepts of *lifting* and *superadditivity* here. The intention of this section is only to motivate the use of *MIR* as a superadditive function when lifting strong valid inequalities to higher dimensions. In terms of network design polyhedra we want to lift facets of sets with a single design variable to strong valid inequalities of the corresponding multi-facility sets (see for instance Section 4.3).

This is only a short overview, we will not go into any details. For a more thorough overview see Nemhauser & Wolsey [1988]. We follow in some parts introductions given by Agra & Constantino [2003] and Atamtürk [2003*a*]. The basic theory of lifting and superadditivity was set up by Wolsey [1976, 1977] and Gu et al. [1999, 2000] and Atamtürk [2004] and others.

We consider the mixed integer set Y as already defined:

$$Y = \{\, (f,x) \in \mathbb{R}^M_+ \times \mathbb{Z}^N_+ : \sum_{j\in M} a_j f_j + \sum_{j \in N} c_j x_j \le d, \quad x_j \le u_j, j \in N \,\}$$

where $u_j \in \mathbb{Z}_+ \backslash \{0\}$ $\forall j \in N$. Let (L, U, R) be a partition of N and set $b := d - c_U^T u_U$. Given $S \subseteq N$ and $h \in \mathbb{R}$ define

$$Y_S(h) := \{\, (f, x_S) \in \mathbb{R}^M_+ \times \mathbb{Z}^S_+ : a^T f + c_S^T x_S \le h, \quad x_j \le u_j, j \in S \,\}.$$

Hence $Y_R(b)$ is the restriction of Y, obtained by setting all variables in L to their lower bound zero and all variables in U to their upper bound u_j. Assume $Y_R(b) \ne \emptyset$.

Let

$$\gamma^T f + \beta_R^T x_R \le \pi \tag{3.11}$$

be a valid inequality for $Y_R(b)$, where γ, β_R are vectors of appropriate dimension and $\pi \in \mathbb{R}$. The lifting problem is now to find a vector $(\beta_L, \beta_U) \in \mathbb{R}^L \times \mathbb{R}^U$ such that

$$\gamma^T f + \beta_R^T x_R + \beta_L^T x_L + \beta_U^T (u_U - x_U) \le \pi$$

is valid for Y.

To compute these coefficients we make use of the so-called *lifting function*, which is associated with (3.11). For $z \in \mathbb{R}^m$ (with $Y_R(b-z) \neq \emptyset$) define

$$\phi_R(z) = \min\{\, \pi - \gamma^T f - \beta_R^T x_R :\ (f, x_R) \in Y_R(b-z)\,\}.$$

($\phi_R(z)$ is a finite value if $Y_R(b-z) \neq \emptyset$, see Atamtürk [2004]). One way to lift variables is to introduce them one by one in a certain sequence (sequential lifting). This leads to an optimisation problem in each of the steps. Suppose $x_i, i \in L$ is the first variable to be lifted. A lifting coefficient β_i produces a valid inequality for $Y_{R\cup\{i\}}$ if and only if the following condition holds ([Wolsey, 1976]):

$$\beta_i x_i \leq \phi_R(c_i x_i) \qquad \forall x_i \geq 1, (f, x_{R\cup\{i\}}) \in Y_{R\cup\{i\}}(b) \tag{3.12}$$

equivalent to

$$\beta_i \leq \min\{\, \frac{\phi_R(c_i x_i)}{x_i} :\ x_i \geq 1, (f, x_{R\cup\{i\}}) \in Y_{R\cup\{i\}}(b)\,\} =: \phi_R^i(c_i).$$

Calculating $\phi_R^i(c_i)$ is a nonlinear optimisation problem in general. However, lifting a binary variable requires the solution of a linear mixed integer program since then $\phi_R^i(c_i) = \phi_R(c_i)$.

Proposition 3.12 (Wolsey [1976]) *If* (3.11) *is valid for* $Y_R(b)$ *and* $-\infty < \beta_i \leq \phi_R^i(c_i)$ *then*

$$\gamma^T f + \beta_R^T x_R + \beta_i x_i \leq \pi \tag{3.13}$$

is valid for $Y_{R\cup\{i\}}(b)$*. Moreover, if* $-\infty < \beta_i = \phi_R^i(c_i)$ *and* (3.11) *defines a* k*-dimensional face of* $\operatorname{conv}(Y_R)$ *then* (3.13) *defines a face of* $\operatorname{conv}(Y_{R\cup\{i\}})$ *of dimension at least* $k+1$.

Note that lifting of variables in U can be done in a similar way by using ϕ_R.
We say that lifting is *exact* if $\beta_i = \phi_R^i(c_i)$. It is known that $\phi_{R\cup\{k\}}^i(c_i) \geq \phi_R^i(c_i)$. For a particular $i \in L$, the later x_i is introduced to the inequality in a lifting sequence, the smaller $\phi_R^i(c_i)$ is, implying that the lifted inequalities may depend on the lifting sequence.

Now suppose that the lifting function ϕ_R is superadditive. In this case, it turns out that lifting is sequence independent and that we can lift all variables in L, U simultaneously. Wolsey [1976] and Gu et al. [2000] show that superadditive lifting functions lead to sequence independent lifting for sets with only binary variables and for mixed 0-1 sets respectively. Atamtürk [2004] proves this for general mixed integer sets.

Proposition 3.13 (Atamtürk [2004]) *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a superadditive function with* $\varphi \leq \phi_R$*. If* (3.11) *is valid for* $Y_R(b)$ *then the lifted inequality*

$$\gamma^T f + \beta_R^T x_R + \sum_{i\in L} \varphi(c_i) x_i + \sum_{i\in U} \varphi(-c_i)(u_i - x_i) \leq \pi \tag{3.14}$$

is valid for Y*. Moreover, if* $\varphi = \phi_R$ *then* $\phi_R(c_i) = \phi_R^i(c_i)$ *and* (3.14) *defines a face of* $\operatorname{conv}(Y)$ *of dimension at least* $k + |L| + |U|$ *if* (3.11) *is a* k*-dimensional face of* $\operatorname{conv}(Y_R(b))$.

We say that φ is a *superadditive lower bound on* ϕ_R and a *valid lifting function* that can be *used for lifting*. Having $\geq$-base inequalities we speak of *subadditive upper bounds* on the lifting function.

Knowing a superadditive lower bound φ on the exact lifting function is a great convenience from a computational point of view. For calculating the lifting coefficients we only have to evaluate $\varphi(c_i)$

for all $i \in L$ and $\varphi(-c_i)$ for all $i \in U$ instead of solving a potentially hard optimisation problem in each step of the lifting procedure.

It is now of interest under which conditions the superadditive *MIR*-function F_d (or the subadditive *MIR*-function G_d) can be used for lifting. An answer to this question can be found in Louveaux & Wolsey [2003].

Proposition 3.14 (Louveaux & Wolsey [2003]) *Suppose the initial valid inequality for $Y_R(b)$ is of the form*

$$\sum_{j \in M} \overline{F}(a_j) f_j + \sum_{j \in R} F(c_j) x_j \le F(d).$$

with F superadditive and nondecreasing. Then $\hat{F} : \mathbb{R} \to \mathbb{R}, \quad \hat{F}(u) = F(d) - F(d-u)$ is a valid lifting function with $\hat{F} \le \phi_R$. If moreover $F = F_d$, then $\hat{F} = F_d$ and hence the MIR*-function itself can be used for lifting.*

Note that $\hat{F}(0) = 0$. We have already stated that superadditive functions with that property can produce valid inequalities for Y (see Remark 3.6). That the *MIR*-function F_d produces a valid inequality is in fact a direct consequence of Theorem 3.2. The last sentence of Proposition 3.14 only says that we can see *MIR* as lifting with a superadditive lower bound on the exact lifting function. *MIR* as a lifting procedure was successfully used in Marchand & Wolsey [1998, 1999] and Louveaux & Wolsey [2003] and Atamtürk [2004].

Example 3.9 (continued) *We have already defined the set $Y^{\ge}(u)$. Now consider the restriction*

$$Y^{\ge}_{1,2}(u) = \{\, x \in \mathbb{Z}^2_+ : \; 4x_1 + 7x_2 \ge 13, \quad x_i \le u, i \in \{1,2\} \,\}$$

*by setting $x_3 = x_4 = 0$. The $\frac{1}{4}$-*MIR *inequality*

$$\mathcal{G}_{13,4}(4)x_1 + \mathcal{G}_{13,4}(7)x_2 \ge \mathcal{G}_{13,4}(13) \qquad \Longleftrightarrow \qquad x_1 + 2x_2 \ge 4$$

*defines a facet of both $\mathrm{conv}(Y^{\ge}_{1,2}(2))$ and $\mathrm{conv}(Y^{\ge}_{1,2}(\infty))$, while the $\frac{1}{7}$-*MIR *inequality*

$$\mathcal{G}_{13,7}(4)x_1 + \mathcal{G}_{13}(7)x_2 \ge \mathcal{G}_{13,7}(13) \qquad \Longleftrightarrow \qquad 4x_1 + 6x_2 \ge 12$$

defines a face of $\mathrm{conv}(Y^{\ge}_{1,2}(2))$ and $\mathrm{conv}(Y^{\ge}_{1,2}(\infty))$ of dimension 0 with the unique point $(0,2)$ satisfying $4x_1 + 6x_2 \ge 12$ with equality.

We now want to lift these inequalities to valid inequalities for the sets $Y^{\ge}(2)$ and $Y^{\ge}(\infty)$. From Proposition 3.14 we know that we can use MIR *as a valid lifting function and from Proposition 3.13 that we can lift simultaneously. The $\frac{1}{4}$-*MIR *inequality for $Y^{\ge}(u)$ is*

$$x_1 + 2x_2 + x_3 + x_4 \ge 4$$

which defines a facet of $\mathrm{conv}(Y^{\ge}(\infty))$ but it defines a face of $\mathrm{conv}(Y^{\ge}(2))$ of dimension only 2. Hence the lifting function of $x_1 + 2x_2 \ge 4$ for $Y^{\ge}_{1,2}(\infty)$ is superadditive and equals the MIR*-function, whereas lifting for $Y^{\ge}_{1,2}(2)$ with* MIR *is valid but not exact.*

Let us calculate the exact lifting coefficients for lifting $-x_1 - 2x_2 \le -4$ to a facet-defining inequality of $Y^{\ge}_{1,2}(2)$ in the sequence $3, 4$.

$$\phi^i_{1,2}(c_3 = -1) = \min\{\, \frac{-4 + x_2 + 2x_2}{x_3} \; : \; x_3 \ge 1, \; -4x_1 - 7x_2 - x_3 \ge -13, \; x_i \le 2 \,\} = \tfrac{-1}{2}$$

It follows that $x_1 + 2x_2 + \frac{1}{2}x_3 \geq 4$ *is facet defining for* $\text{conv}(Y^{\geq}_{1,2,3}(2))$. *Similarly,* $\phi^i_{1,2,3}(c_4 = -1) = -1$ *resulting in the facet-defining inequality*

$$x_1 + 2x_2 + \tfrac{1}{2}x_3 + x_4 \geq 4$$

for $\text{conv}(Y^{\geq}(2))$. *(It turns out, that the exact lifting function is superadditive as well, so the sequence* $4, 3$ *produces the same inequality dominating the* $\frac{1}{4}$-MIR *inequality.)*

The $\frac{1}{7}$-MIR *inequality for* $Y^{\geq}(u)$ *is* $4x_1 + 6x_2 + x_3 + 2x_4 \geq 12$ *which defines a facet of both* $\text{conv}(Y^{\geq}(\infty))$ *and* $\text{conv}(Y^{\geq}(2))$. *Superadditive lifting with* MIR *is exact in both cases and surprisingly the lifted inequality is facet-defining although* $4x_1 + 6x_2 \geq 12$ *only defined a lower dimensional face of* $\text{conv}(Y^{\geq}_{1,2}(\infty))$ *and* $\text{conv}(Y^{\geq}_{1,2}(2))$.

Summary In this section we have emphasised the usefulness of *MIR* as a valid superadditive lifting function. Gu et al. [1999, 2000], Atamtürk [2003*a*,*b*], Louveaux & Wolsey [2003] and Agra & Constantino [2003] and many others, construct superadditive lower bounds on exact lifting functions different to the simple *MIR*-functions considered in this thesis.

The intention of the author is to show that for network design problems and the sets considered in this thesis it suffices to consider *MIR*. This restriction at least provides the possibility of developing a generic separation procedure that is able to detect different classes of robust and strong valid inequalities and that might be useful for practical implementations.

3.3 Upper bounds, *complemented MIR* inequalities, covers and packs

In Example 3.9 we observed that *MIR*-inequalities might be weak, if variables are bounded. In the sequel a procedure is motivated that exploits the special structure of such sets. Again consider the set

$$Y = \{\, (f, x) \in \mathbb{R}^M_+ \times \mathbb{Z}^N_+ : \sum_{j \in M} a_j f_j + \sum_{j \in N} c_j x_j \leq d, \quad x_j \leq u_j \,\}$$

with $u_j \in \mathbb{Z}_+ \backslash \{0\}, j \in N$. A basic idea now is that of *complementing*. Let $U \subseteq N$, $R := N \backslash U$ and define $\bar{x}_j := u_j - x_j$ for $j \in U$. The base inequality (3.1) can be rewritten as

$$\sum_{j \in M} a_j f_j + \sum_{j \in R} c_j x_j + \sum_{j \in U} -c_j \bar{x}_j \leq d - \sum_{j \in U} c_j u_j =: b.$$

Since $\bar{x}_j \in \mathbb{Z}_+$ for all $j \in U$, it is straightforward to apply *MIR* (Theorem 3.2) now, which after reintroducing the original variables, results in

$$\sum_{j \in M} \overline{F}_b(a_j) f_j + \sum_{j \in R} F_b(c_j) x_j + \sum_{j \in U} F_b(-c_j)(u_i - x_j) \leq F_b(b). \tag{3.15}$$

MIR inequalities of type (3.15) will be called *complemented-MIR inequalities*. These inequalities were introduced by Marchand [1997] and Marchand & Wolsey [1998] (see Section 3.4). In fact we already know them. Considering the restriction Y_R of Y obtained by fixing variables in U to their upper bound we get

$$\sum_{j \in M} \overline{F}_b(a_j) f_j + \sum_{j \in R} F_b(c_j) x_j \leq F_b(b). \tag{3.16}$$

as a valid *MIR*-inequality for the set $Y_R(b)$. Now using the *MIR*-function F_b as a superadditive valid lifting function, we can lift all variables in U simultaneously and arrive at (3.15) (see Proposition 3.13 and Proposition 3.14). Similarly,

$$\sum_{j \in M} \overline{F}_b(a_j) f_j + \sum_{j \in U} F_b(-c_j)(u_i - x_j) \leq F_b(b). \tag{3.17}$$

is a valid complemented-*MIR* inequality for the restriction $Y_U(d)$ obtained by fixing variables in R to their lower bound zero. Lifting with F_b yields (3.15) again.

It follows that the three procedures

1. Complement variables in U and apply *MIR*,
2. Fix variables in R to their lower bound, apply (complemented-)*MIR* and finally lift variables in R using the superadditive function F_b,
3. Fix variables in U to their upper bound, apply *MIR* and finally lift variables in U using the superadditive function F_b

are equivalent. This observation is important because it allows us to make a statement about the strength of the inequality (3.15). If (3.16) is facet-defining for $\text{conv}(Y_R(b))$ and F_b is the exact lifting function for lifting variables in U, then (3.15) is facet-defining for $\text{conv}(Y)$. Similarly, if (3.17) is facet-defining for $\text{conv}(Y_U(d))$ and F_b is the exact lifting function for lifting variables in R, then (3.15) is facet-defining for $\text{conv}(Y)$. If otherwise F_b is not the exact lifting function in both cases, we can at least use it for (computationally easy) simultaneous lifting and might get high dimensional faces for $\text{conv}(Y)$.

It turns out that it is crucial to choose the set U in such a way that the restricted inequalities ((3.16) or (3.17)) define facets for the restricted sets $\text{conv}(Y_R(b))$ or $\text{conv}(Y_U(d))$. It is in this context that the terms and definitions of *covers* and *packs* arise.

In the sequel let the set Y be given with $c_j > 0$ for all $j \in N$.

Definition 3.15 *A subset C of N is called a* cover *if $u_C^T c_C = \sum_{j \in C} u_j c_j > d$. Set $\lambda := u_C^T c_C - d$. Similarly, a subset P of N is called a* pack *if $\mu := d - u_P^T c_P > 0$.*

Covers and packs and the corresponding *cover*- and *pack-inequalities* have been studied extensively in the literature. The main approach here is that of fixing variables with respect to an appropriately chosen cover or pack, considering a facet-defining inequality of the convex hull of the restriction of Y obtained this way and then lifting this facet to a high dimensional face of $\text{conv}(Y)$. See [Atamtürk, 2003*a*] for a detailed introduction and literature overview.

Marchand & Wolsey [1998] show that some classes of lifted cover- and pack inequalities are just *MIR*-inequalities, or are even dominated by *MIR*-inequalities. In the following we will give two examples of this.

Covers First we consider a *mixed 0-1 knapsack set*, that is Y but with $u_j = 1$ and $c_j > 0$ for all $j \in N$. Let C be a cover with excess λ such that $\lambda \leq \bar{c} := \max_{j \in C} c_j$. Set $R := N \setminus C$. Fixing all variables in R to zero and complementing all variables in C results in

$$\sum_{j \in M} a_j f_j + \sum_{j \in C} -c_j \bar{x}_j \leq -\lambda.$$

Now calculating the $\frac{1}{\bar{c}}$-(complemented)-*MIR* inequality gives

$$
\begin{aligned}
& \sum_{j\in M} \overline{\mathcal{F}}_{-\lambda,\bar{c}}(a_j)f_j + \sum_{j\in C} \mathcal{F}_{-\lambda,\bar{c}}(-c_j)(1-x_j) \le \mathcal{F}_{-\lambda,\bar{c}}(-\lambda) \\
\iff & \sum_{j\in M} a_j^- f_j \qquad\quad + \sum_{j\in C} \mathcal{F}_{-\lambda,\bar{c}}(-c_j)(1-x_j) \le -\lambda \\
\iff & \sum_{j\in M} a_j^- f_j \qquad\quad - \sum_{j\in C} \min(\lambda, c_j)(1-x_j) \le -\lambda. \qquad (3.18)
\end{aligned}
$$

The last step follows from Lemma 3.11 iv).

(3.18) is the so-called *mixed 0-1 cover inequality* (Marchand & Wolsey [1999]) and defines a facet of $\mathrm{conv}(Y_C)$. If C is a minimal cover, i. e. $c_j \ge \lambda$ for all $j \in C$ and also $M = \emptyset$, then inequality (3.18) reduces to

$$\sum_{j\in C} x_j \le |C| - 1,$$

the well-known *cover inequality* for 0-1 knapsack sets. (Note that for mixed 0-1 knapsack sets the minimality of a cover is not a necessary condition for (3.18) to be facet-defining, it suffices to have $\lambda \le \bar{c}$.) It is now obvious that we can use $\mathcal{F}_{-\lambda,\bar{c}}$ to lift inequality (3.18).

$$\sum_{j\in M} a_j^- f_j + \sum_{j\in R} \mathcal{F}_{-\lambda,\bar{c}}(c_j)x_j - \sum_{j\in C} \min(\lambda, c_j)(1-x_j) \le -\lambda \qquad (3.19)$$

is valid for Y and it can be a strong. See Marchand & Wolsey [1999] and Atamtürk [2003*a*] for exact lifting functions and superadditive lower bounds different from $\mathcal{F}_{-\lambda,\bar{c}}$.

Packs Now, given Y with $c_j > 0, j \in N$, let P be a pack with residual μ such that $\mu < \bar{c} := \max_{j\in N\setminus P} c_j$. Set $R := N\setminus P$. Now complementing all variables in P results in

$$\sum_{j\in M} a_j f_j + \sum_{j\in P} -c_j\bar{x}_j + \sum_{j\in R} c_j x_j \le \mu.$$

Calculating the $\frac{1}{\bar{c}}$-(complemented)-*MIR* inequality gives

$$\sum_{j\in M} a_j^- f_j + \sum_{j\in P} \mathcal{F}_{\mu,\bar{c}}(-c_j)(u_j - x_j) + \sum_{j\in R} \mathcal{F}_{\mu,\bar{c}}(c_j)x_j \le 0. \qquad (3.20)$$

We call (3.20) a (lifted) *mixed integer pack inequality*. Using Lemma 3.11 iv) gives $\mathcal{F}_{\mu,\bar{c}}(c_j) = (c_j - \mu)^+$ for $j \in R$ since $\bar{c} > \mu$ and $\bar{c} \ge c_j$. From Corollary 3.8 we know that $\mathcal{F}_{\mu,\bar{c}}(-c_j) \ge -c_j$. It follows that (3.20) is at least as strong as

$$
\begin{aligned}
& \sum_{j\in M} a_j^- f_j \ + \sum_{j\in P} -c_j(u_j - x_j) + \sum_{j\in R} (c_j - \mu)^+ x_j \le 0 \\
\iff & \sum_{j\in M} a_j^- f_j \ + \sum_{j\in P} c_j x_j \qquad\quad + \sum_{j\in R} (c_j - \mu)^+ x_j \le \sum_{j\in P} c_j u_j = d - \mu,
\end{aligned}
$$

which Martin & Weismantel [1997] and Weismantel [1997] refer to as a *weight inequality*. Weight inequalities define facets of Y under certain conditions. Inequality (3.20) reduces to

$$\sum_{j \in R} (c_j - \mu)^+ x_j \leq 0,$$

the well-known *pack inequality* for 0-1 knapsack sets, if $M = \emptyset$ and variables in the pack P are fixed to their upper bound 1.

Example 3.9 (continued) *We have already considered the set*

$$Y^{\geq}(2) = \{ \, x \in \mathbb{Z}_+^4 : \; 4x_1 + 7x_2 + x_3 + 2x_4 \geq 13, \quad x_i \leq 2, i \in \{1, .., 4\} \, \}$$

and derived strong valid inequalities by MIR. *Here, we try to obtain new facets by considering covers and packs. We only state five examples. Most of the nontrivial facet-defining inequalities for $Y^{\geq}(2)$ can be obtained with this procedure. First we will only consider covers and packs with excess or residual smaller than $\bar{c} := \max(c_j)_{j \in N} = 7$. We have not defined covers and packs for $\geq$-inequalities yet. But by complementing all variables we arrive at the equivalent system*

$$Y^{\leq}(2) = \{ \, \bar{x} \in \mathbb{Z}_+^4 : \; 4\bar{x}_1 + 7\bar{x}_2 + \bar{x}_3 + 2\bar{x}_4 \leq 15, \quad \bar{x}_i \leq 2, i \in \{1, .., 4\} \, \}.$$

$C = \{2, 3\}$ *is a cover with excess* $\lambda = 7 \cdot 2 + 1 \cdot 2 - 15 = 1$. *Complementing all variables in the cover yields*

$$4\bar{x}_1 - 7x_2 - x_3 + 2\bar{x}_4 \leq -1 \qquad \Longleftrightarrow \qquad -4\bar{x}_1 + 7x_2 + x_3 - 2\bar{x}_4 \geq 1.$$

We arrive at the same inequality by defining $P = \{1, 4\}$ to be a pack with respect to the original $\geq$-base inequality and by complementing all variables in the pack. Now calculate the $\frac{1}{\bar{c}}$-MIR *inequality given by*

$$x_2 + x_3 \geq 1.$$

Considering $C = \{2, 3, 4\}$ gives a cover with $\lambda = 5$ and the MIR-*cover inequality*

$$2x_1 + 5x_2 + x_3 + 2x_4 \geq 9.$$

Now let $P = \{1, 4\}$. It follows that $\mu = 15 - 4 \cdot 2 - 2 \cdot 2 = 3$. Complementing all variables in the pack gives

$$-4x_1 + 7\bar{x}_2 + \bar{x}_3 - 2x_4 \leq 3 \qquad \Longleftrightarrow \qquad 4x_1 - 7\bar{x}_2 - \bar{x}_3 + 2x_4 \geq -3.$$

The same inequality is obtained by defining $C = \{2, 3\}$ to be a cover with respect to the original $\geq$-base inequality and by complementing all variables in the cover. The $\frac{1}{\bar{c}}$-MIR *inequality is*

$$4x_1 + 4x_2 + 2x_4 \geq 8.$$

Considering $P = \{1, 3\}$ gives a pack with $\mu = 5$ and the MIR- *pack inequality*

$$2x_1 + 2x_2 + x_3 \geq 4.$$

All of the cover- and pack inequalities we have derived in this way define facets of $Y^{\geq}(2)$*. It is crucial to only consider covers and packs with small values of* λ *and* μ*. Let for instance* $P = \{3, 4\}$ *with* $\mu = 9$*. Complementing gives*

$$4\bar{x}_1 + 7\bar{x}_2 - x_3 - 2x_4 \leq 9 \quad \Longleftrightarrow \quad -4\bar{x}_1 - 7\bar{x}_2 + x_3 + 2x_4 \geq -9.$$

Choosing $\bar{c} = 10 > \mu$ *leads to the* $\frac{1}{\bar{c}}$-MIR *inequality*

$$x_3 + x_4 \geq 0,$$

which is trivial and does not define a facet of $Y^{\geq}(2)$*.*

Summary In this section it has been shown how to exploit the special structure of bounded mixed integer sets. We simply complemented variables in previously chosen covers and packs before scaling and *MIR*. In this context it is important that the excess λ for covers or the residual μ for packs is small with respect to the coefficients of the base inequality. Motivated by the stated strong valid cover and pack inequalities and by Example 3.9, we will restrict our attention to *MIR*-cover- and *MIR*-pack inequalities with

$$\bar{c} = \max(c_j)_{j \in N} > \lambda, \mu,$$

where $\frac{1}{\bar{c}}$ will be used as the factor to scale the base inequalities.

3.4 A *MIR* procedure

Marchand [1997] and Marchand & Wolsey [1998] observed that many families of strong valid inequalities of certain mixed integer sets are in fact *MIR*-inequalities obtained by the following procedure:

1. **Aggregation:** Choose a positive linear combination of the inequalities that describe the mixed integer set to get a valid base inequality of the form (3.1) or (3.4).

2. **Bound Substitution:** Substitute continuous variables by potentially given (variable) lower or upper bounds.

3. **Complementing:** Choose a subset U of the integer variables and complement them with respect to their bound constraints.

4. **Scaling:** Divide the base inequality by some positive integer c.

5. ***MIR*:** Apply *MIR*.

Strong valid inequalities with respect to certain mixed integer sets that can be obtained by the procedure above are, for instance:

- (lifted) *cover and pack inequalities* (Atamtürk [2003*a*])

- *arc residual capacity inequalities* (Magnanti et al. [1993])

- *flow cut inequalities* (Bienstock & Günlük [1996] and Chopra et al. [1998] and Atamtürk [2002])
- (lifted) *flow cover* and *flow pack inequalities* (Gu et al. [1999] and Atamtürk [2001])
- *knapsack partition inequalities* (Pochet & Wolsey [1992, 1995])

Given a general mixed integer set, the simple idea of Marchand [1997] and Marchand & Wolsey [1998] is to apply each of the steps *aggregating*, *substituting*, *complementing* and *scaling* heuristically before applying *MIR*. They propose a very generic separation heuristic based on this *MIR* procedure and showed that, when integrated into a Branch & Cut algorithm, it computationally gives results as good as, or better than, those obtained from several existing (general purpose) separation routines.

The approach used throughout this thesis is slightly different. We are not faced with general mixed integers sets but with network design problems and want to take into account the structure of those problems and the underlying networks. In fact, every of the steps above will be part of our *MIR* procedure, but they will not be used heuristically (in the sense of Marchand [1997] and Marchand & Wolsey [1998]) and although they can be useful for implementations, they are of a theoretical nature rather than being a pseudo-code. In the presence of a concrete class of inequalities one has to face the *separation problem*, which usually causes some modifications in actual practice (Chapter 7).

The procedure, that will be described below in more detail, tries to generalise the way to obtain the strong inequalities specific for network design problems, which will be introduced in Chapter 4, 5 and 6. For a thorough description see the appropriate chapter. In Chapter 4 we will see how to obtain certain flow cut inequalities and in Chapter 5 we will consider flow cover and flow pack inequalities. Sections 6.3, 6.2 and 6.4 are devoted to multi cut inequalities, arc residual capacity inequalities and knapsack partition inequalities, respectively.

The following approach may even be of interest for problems different to those considered in this thesis. We concentrate on the DIrected case. The BIdirected and UNdirected cases are analogous.

Aggregation Given a network design polyhedron as defined in Section 2.2, choose subsets of the nodes V, the arcs A (or edges E) and the commodities K and consider a linear combination of the flow conservation constraints (2.2) or (2.3), the capacity constraints (2.4), (2.5), (2.6) and the non-negativity constraints (2.7) or (2.8) with respect to the given subsets. The selection of nodes and arcs (or edges) will not be done heuristically but should reflect a certain structure of the underlying network. For example, it turns out that it is useful to consider

- a subset S of the nodes V and subsets of the arcs in the cut $\delta(S)$ defined by S (see Chapter 4, Chapter 5 and Section 6.4)
- a partition of the network nodes V and the corresponding multi cut (see Section 6.3),
- a single arc (see Section 6.2).

In this context, it is crucial to know more about the polyhedral structure of the relaxations obtained with such an aggregation procedure, such as *cut sets*, *single arc sets* or *multi cut sets*. In Chapter 4 and 5 we consider *cut sets* and learn how to choose subsets of commodities, arcs and nodes such that our approach produces strong valid or even facet-defining inequalities.

By aggregating valid inequalities as mentioned we arrive at a single constraint of the form (3.4).

We could directly calculate a $\frac{1}{c}$-*MIR* inequality with respect to this single constraint. But since $\overline{\mathcal{G}}_{d,c}(a_j) = a_j^+$ (in the only intersting case that $\langle \frac{d}{c} \rangle > 0$), we would round up negative coefficients of flow variables to zero.

Substituting The intention of this step is to obtain a base inequality with non-negative coefficients for all flow variables. Suppose that after aggregation all flow variables f_a^k for $a \in A_S$ and $Q \subseteq K$ have the same negative coefficient in (3.4).

We use the slack variable $\bar{f}_a^Q \geq 0$ of the corresponding capacity (or variable upper bound) constraint

$$f_a^Q + \bar{f}_a^Q = \sum_{t \in T} c^t x_a^t$$

to substitute f_a^Q for $\sum_{t \in T} c^t x_a^t - \bar{f}_a^Q$.

If we are able to bound the flow f_a^Q with some constant value:

$$f_a^Q + \bar{f}_a^Q = u_a^Q \in \mathbb{R}_+,$$

we can similarly substitute f_a^Q for $u_a^Q - \bar{f}_a^Q$. (This is complementing of flow variables and will be used in Section 6.2.)

Proceeding this way for all flow variables with negative coefficients we get a modified base inequality of type (3.4) where all coefficients of flow variables are non-negative.

Complementing We choose a subset U of all bounded integer design variables. The set U will be either empty or an appropriately chosen cover or pack similar to the examples of the last section. In Chapter 5 we consider flow covers and flow packs as an extension to covers and packs. We choose U such that the corresponding excess λ or residual μ is smaller than the maximum $\bar{c}$ of the coefficients of the base inequality.

All variables in U will be complemented.

Scaling and *MIR* It remains to define the factor c that is used to scale the base inequality. For many strong inequalities given in the literature (and all inequalities considered in this thesis) it suffices to select c from the coefficients of the base inequality.

Now we calculate the $\frac{1}{c}$-*MIR* inequality.

Since the $\frac{1}{c}$-*MIR* can be seen as new base inequality the step ‘Scaling and *MIR*’ can be repeated. Finally, we have to restate the resulting inequality in terms of the original variables.

3.5 Summary

This chapter is the basis for the rest of this thesis. All the necessary notation concerning *MIR* has been introduced. It has been shown how to derive numerically safe *MIR* inequalities. We have learned how to scale them such that all coefficients are small integers.

It is often useful to consider restrictions of given mixed integer sets by fixing some variables to their bounds. It was emphasised that *MIR* can be used for the simultaneous lifting of the valid inequalities of such restrictions. It was furthermore shown that if a valid inequality for a restriction has already been obtained by *MIR*, then *MIR* provides a canonic valid superadditive lifting function. In this context the strength of such lifted *MIR* inequalities can be investigated by comparing the *MIR* function with the exact lifting function.

In cases where variables are bounded by some constant value, it has been explained how to exploit this additional information. Those variables can be complemented before applying *MIR*, which is equivalent to fixing those variables to their bound, obtaining a valid inequality for the restriction by *MIR* and then lifting with the same *MIR* function. It turned out that it is crucial to properly choose the sets of variables that are to be complemented. It has been shown that well-known cover (pack) inequalities are obtained with this approach when complementing variables in the corresponding cover (pack). Those covers (packs) have to be chosen such that they are minimal (maximal) in the sense that their excess (residual) is small with respect to the coefficients of the base inequality considered.

A *MIR* procedure has been proposed which is able to produce strong valid inequalities for the network design problems investigated in this thesis. It is important for the success of such a *MIR* procedure to restrict the large pool of possible base inequalities. In the following chapters, especially Chapter 4, we will investigate the facial structure of certain relaxations of network design polyhedra, which will provide more information about how to derive good base inequalities.

Chapter 4

Cut sets and flow cut inequalities

4.1 Introduction

Cut sets (or *single node flow sets*) arise from the aggregation of flow conservation constraints for a node set $S \subset V$. The network design polyhedra are restricted to the cut $\delta(S)$ and the two artificial nodes S and $V \setminus S$. The resulting cut set polyhedra, or simply cut sets, serve as relaxations of the original network design polyhedra. It is important to understand the polyhedral structure of cut sets because they cover a significant part of the characteristics of the related network design polyhedra. When developing Branch & Cut algorithms for those problems, facet-defining inequalities for cut sets play a crucial role (see Chapter 7).

Literature review The cut set polyhedron with bounded integer design variables, and here especially the 0-1 case, has been investigated by many authors. See Chapter 5 for a literature review. In this chapter we consider unbounded design variables but most of the stated results are useful for the bounded case too. In fact, all of them hold if the bounds given are large enough. Valid inequalities for cut sets will be called *cut set inequalities*.

The most important cut set inequality is the *cut inequality*. It simply says that the demand that can be routed across a cut of the network is upper bounded by the installed capacity. See Schrijver [2003, Volume C, Chapter 70] for a survey on cut inequalities and multi-commodity flow problems with existing capacities. Cut inequalities for capacitated network design problems with UNdirected capacity constraints were investigated by Baharona [1994] and in a series of articles by Magnanti & Mirchandani [1993] and Magnanti et al. [1993, 1995]. Bienstock et al. [1995] consider network design polyhedra with DIrected capacity constraints. They present cut inequalities and second class of cut set inequalities, which we will call *simple flow cut inequalities* generalising cut inequalities. The support of such inequalities additionally contains certain flow variables with respect to the edges of the cut considered. Cut inequalities and simple flow cut inequalities for the BIdirected case have been studied by Bienstock & Günlük [1996] The most general form of *flow cut inequality* as investigated in this chapter was first introduced by Chopra et al. [1998]. They consider DIrected supply graphs. Atamtürk [2002] presents a detailed analysis for a cut set with DIrected capacity constraints, which we refer to as CS^{DI}. He states necessary and sufficient conditions for flow cut inequalities to be facet-defining for CS^{DI} even for the general multi-commodity multi-facility case. The work of Atamtürk [2002] can be seen as the basis of this chapter.

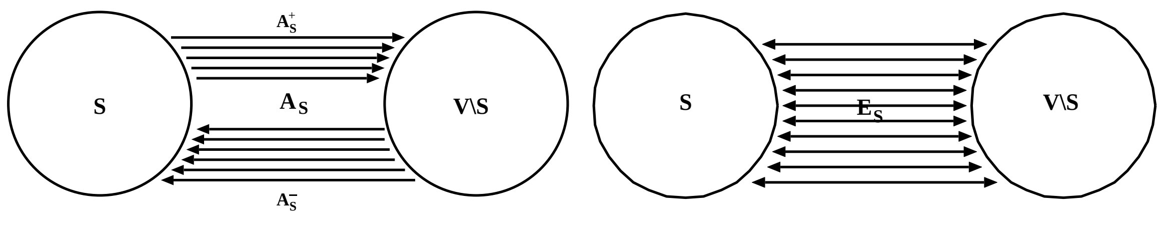
Outline of this chapter After introducing the cut sets CS^{DI}, CS^{BI} and CS^{UN} and stating some additional definitions and assumptions, we will motivate the analysis of cut sets by showing that facet-defining inequalities for CS^{DI}, CS^{BI} or CS^{UN} define facets of NDP^{DI}, NDP^{BI} or NDP^{UN} with certain additional demands on the structure of the underlying graphs.

In Section 4.2 we will first restrict ourselves to single facility problems and will investigate the polyhedral structure of cut sets in detail. We will present facet-proofs for certain classes of flow cut inequalities. They will be established in a very general form for all three capacity models. For the cut sets with undirected supply graphs CS^{BI} and CS^{UN} this is done for the first time, generalising inequalities proposed by Magnanti & Mirchandani [1993] and Bienstock & Günlük [1996]. It will be emphasised that flow cut inequalities can be obtained by *MIR*.

We start with cut sets for directed supply graphs in Section 4.2.1. The most important results of Atamtürk [2002] for CS^{DI} will be summarised and even supplemented. In Section 4.2.2 the BIdirected and UNdirected versions of cut sets will be studied. We will define a class of flow cut inequalities, similar to that of Chopra et al. [1998] for CS^{DI}, which contains known flow cut inequalities, so-called cut inequalities and simple flow cut inequalities, as a special case. The corresponding facet-proofs extend results of Magnanti & Mirchandani [1993] and Bienstock & Günlük [1996]. As an extension, a new class of facet-defining cut set inequalities is stated that has no analogue for the DIrected case.

In Section 4.3 we will investigate how all those facet-defining inequalities for cut sets with a single facility can be generalised to strong valid inequalities for the multi-facility case. Again, the results for directed supply graphs are from Atamtürk [2002]. We will show that the exact lifting function he uses to lift valid flow cut inequalities of single facility restrictions is in fact the *MIR* function introduced in Chapter 3. We will make use of the same *MIR* function for the cut sets CS^{BI} and CS^{UN} and propose a *MIR* procedure to obtain strong valid flow cut inequalities in the general multi-facility, multi-commodity case for all three capacity models.

One intention of this chapter is to present strong valid flow cut inequalities in a closed form for all three capacity models, including all the special cases. We will elaborate on the differences between the polyhedral structures of CS^{DI}, CS^{BI} and CS^{UN}.



(i) A cut based on a directed supply graph (ii) A cut based on an undirected supply graph

Figure 4.1: Cuts and flow directions

Definitions We will now define the three cut sets corresponding to the different capacity models. Let $A_S := \delta(S) \neq \emptyset$ be a dicut in the digraph $G = (V, A)$ where $\emptyset \neq S \subset V$. Set $A_S^+ := \delta^+(S)$ and $A_S^- := \delta^-(S)$. For every commodity respectively we sum up all flow conservation constraints (2.2) for $i \in S$ and arrive at the following system of inequalities.

$$\sum_{a \in A_S^+} f_a^k - \sum_{a \in A_S^-} f_a^k = d_S^k \qquad \forall k \in K \tag{4.1}$$

$$\sum_{k \in K} f_a^k \leq \sum_{t \in T} c^t x_a^t \qquad \forall a \in A_S \tag{4.2}$$

$$0 \leq f_a^k, x_a^t \qquad \forall a \in A_S, k \in K, t \in T, \tag{4.3}$$

where $d_S^k := \sum_{i \in S} d_i^k, k \in K$ and $c \in \mathbb{Z}_+ \backslash \{0\}$. The capacities $c^t, t \in T$ were introduced in Section 2.2. The corresponding multi-commodity, multi-facility **cut set polyhedron**, or simply **cut set**, for directed supply graphs is defined as:

$$CS^{DI} := \text{conv}\{\, (f, x) \in \mathbb{R}^{|K||A_S|} \times \mathbb{Z}^{|A_S||T|} : \ (f, x) \text{ satisfies (4.1), (4.2) and (4.3)} \,\}$$

For undirected supply graphs $G = (V, E)$ let $E_S := \delta(S) \neq \emptyset$ be a cut with $S \subset V$. Aggregating the flow conservation constraints (2.3) as above gives

$$\sum_{e=ij \in E_S} f_{ij}^k - \sum_{e=ij \in E_S} f_{ji}^k = d_S^k \qquad \forall k \in K \tag{4.4}$$

$$\begin{aligned} \sum_{k \in K} f_{ij}^k &\leq \sum_{t \in T} c^t x_e^t \qquad \forall e = ij \in E_S \\ \sum_{k \in K} f_{ji}^k &\leq \sum_{t \in T} c^t x_e^t \qquad \forall e = ij \in E_S \end{aligned} \tag{4.5}$$

$$\sum_{k \in K} (f_{ij}^k + f_{ji}^k) \leq \sum_{t \in T} c^t x_e^t \qquad \forall e = ij \in E_S \tag{4.6}$$

$$0 \leq f_{ij}^k, f_{ji}^k, x_e^t \ \forall e = ij \in E_S, k \in K, t \in T \tag{4.7}$$

The corresponding multi-commodity, multi-facility **cut sets** for undirected supply graphs are:

$$CS^{BI} := \text{conv}\{\, (f, x) \in \mathbb{R}^{2|K||E_S|} \times \mathbb{Z}^{|E_S||T|} : \ (f, x) \text{ satisfies (4.4), (4.5) and (4.7)} \,\}$$
$$CS^{UN} := \text{conv}\{\, (f, x) \in \mathbb{R}^{2|K||E_S|} \times \mathbb{Z}^{|E_S||T|} : \ (f, x) \text{ satisfies (4.4), (4.6) and (4.7)} \,\}$$

As for NDP^{DI}, NDP^{BI} and NDP^{UN} we do not write CS^{DI}, CS^{BI} and CS^{UN} as functions of S or any other parameter. The cut sets are defined in the space of cut variables but it is obvious that

$$NDP^{DI} \subseteq \overline{CS}^{DI} := CS^{DI} \times \mathbb{R}^{|K|(|A|-|A_S|)} \times \mathbb{R}^{|T|(|A|-|A_S|)},$$

since a point $(f, x) \in NDP^{DI}$ satisfies (4.2) and (4.3) by definition and (4.1) is the sum of equations satisfied by (f, x) for every $k \in K$. Hence every valid inequality for $\overline{CS}^{DI}$ is valid for NDP^{DI}.

In the sequel, $\overline{CS}^{DI}$ will not be mentioned anymore, instead we say that a valid inequality for CS^{DI} (in the space of the dicut-variables) is valid for NDP^{DI} (in the space of the original variables). Similarly, we say that a valid inequality for CS^{BI} (CS^{UN}) is valid for NDP^{BI} (NDP^{UN}).

Valid inequalities for CS^{DI}, CS^{BI} and CS^{UN} will be called **cut set inequalities**.

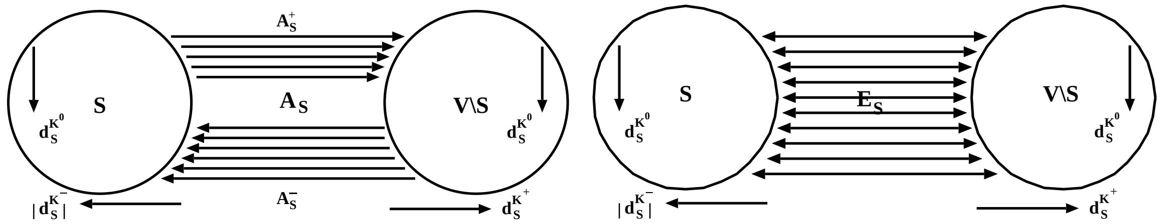
Note that (2.1) implies

$$d_S^k = -d_{V\setminus S}^k.$$

Thus cut sets, together with an objective function, can be seen as two-node network design problems (see Figure 4.1). For every commodity, a flow on the cut or dicut has to be established such that the (aggregated) demand d_S^k is satisfied. Moreover, the cut sets of S and $V\setminus S$ are identical, since we only have to multiply every flow conservation constraint by -1.

Given $k \in K$ we will distinguish the direction of the demand d_S^k that has to be routed across the cut. Define

$$\begin{aligned} \boldsymbol{K^+} &:= \{\, k \in K : \; d_S^k > 0 \,\} \\ \boldsymbol{K^-} &:= \{\, k \in K : \; d_S^k < 0 \,\} \\ \boldsymbol{K^0} &:= \{\, k \in K : \; d_S^k = 0 \,\} \end{aligned}$$



(i) A cut based on a directed supply graph

(ii) A cut based on an undirected supply graph

Figure 4.2: Cuts and demand directions

Hence $K = K^+ \cup K^- \cup K^0$. Commodities $k \in K^+$ are called **positive commodities** and those in K^- are called **negative commodities** (see Figure 4.2). Note that given a commodity $k \in K^0$, the corresponding demands can be satisfied without the need for flow crossing the cut if both $G[S]$ and $G[V\setminus S]$ are connected for undirected supply graphs or strongly connected for directed supply graphs, because the problem of finding a feasible flow for k can then be restricted to these subgraphs.

Note that these commodity sets depend on the chosen network cut. For simplicity we omit the subscript S.

Additional Assumptions The case $K^- \cup K^+ = \emptyset$ is not interesting and since we can switch between the cut sets of S and $V\setminus S$, we assume that there is at least one positive commodity: $K^+ \neq \emptyset$.

For simplicity we make the following additional assumptions, which will be used throughout the rest of this thesis. For directed supply graphs and CS^{DI} we claim that

- $A_S^+ \neq \emptyset$,
- if $K^- \neq \emptyset$ then $A_S^- \neq \emptyset$ and
- if $K^0 \neq \emptyset$ then both A_S^+ and A_S^- are not empty.

For the cut sets CS^{BI} and CS^{UN} it is required that $E_S \neq \emptyset$ as already mentioned.

Before we investigate the dimension of cut sets we show that for the cut set CS^{UN} we can assume $K^- = \emptyset$, w. l. o. g.. Commodities for cut sets with UNdirected capacity constraints can always be seen as being undirected (neither positive nor negative). Consider CS^{UN} with $|K^+| > 0$, $|K^-| > 0$. We will construct an identical cut set with only non-negative commodities by simply renaming flow variables. It follows that the direction of the demands for UNdirected problems is just a matter of modelling. Consider the following new flow vector $\widetilde{f}$:

$$\widetilde{f}^k_{ij} := f^k_{ij} \quad \text{and} \quad \widetilde{f}^k_{ji} := f^k_{ji} \qquad \forall e = ij \in E_S, k \in K \setminus K^-$$
$$\widetilde{f}^k_{ij} := f^k_{ji} \quad \text{and} \quad \widetilde{f}^k_{ji} := f^k_{ij} \qquad \forall e = ij \in E_S, k \in K^-$$

What we have done is change the direction of demands in K^- by swapping the corresponding flow variables. By additionally multiplying the flow conservation constraints by -1 for all negative commodities, we have transformed K^- into a set of positive commodities. Given $k \in K^-$, the corresponding flow conservation constraint is now written:

$$\sum_{e=ij \in E_S} \widetilde{f}^k_{ij} - \sum_{e=ij \in E_S} \widetilde{f}^k_{ji} = -d^k_S > 0.$$

That the cut set defined for $(\widetilde{f}, x)$ and the cut set CS^{UN} are identical, follows from the fact that the capacity constraints are identical since

$$\sum_{k \in K} (\widetilde{f}^k_{ij} + \widetilde{f}^k_{ji}) = \sum_{k \in K} (f^k_{ij} + f^k_{ji}).$$

The latter is not true for CS^{BI}. Here we explicitly make use of the UNdirected formulation and the capacity constraints (4.6).

In the following, we will assume $K^- = \emptyset$ whenever referring to CS^{UN} unless explicitly stated otherwise.

Dimension of cut sets It is obvious that the dimension of CS^{DI} is at most $|K||A_S| + |T||A_S| - |K|$ since there are $|K||A_S| + |T||A_S|$ variables and $|K|$ equations satisfied by every point in CS^{DI}. Similarly, the dimension of CS^{BI} and CS^{UN} is at most $2|K||E_S| + |T||E_S| - |K|$.

Under the assumptions above there are no additional implied equations and we can formulate the following lemmas:

Lemma 4.1 (Atamtürk [2002]) *The dimension of* CS^{DI} *is exactly* $|K||A_S| + |T||A_S| - |K|$.

Some of our claims are even necessary for this lemma to hold. If $A^+_S = \emptyset$ we cannot route the demand for K^+. Similarly, we cannot route the demand for $K^- \neq \emptyset$ if $A^-_S = \emptyset$. In both cases the polyhedron CS^{DI} is empty. If on the other hand $K^0 \neq \emptyset$ and either A^+_S or A^-_S is empty, then the flow for every $k \in K^0$ has to be fixed to zero on every arc of the cut, which gives additional implied equations if $|A_S| > 1$.

Lemma 4.2 $CS^{UN} \subseteq CS^{BI}$

Proof. A point $p = (f, x)$ in CS^{UN} satisfies (4.4), (4.6) and (4.7). But from (4.6) and (4.7) follows (4.5) and thus $p \in CS^{BI}$. ■

Lemma 4.3 (Magnanti et al. [1995], Bienstock & Günlük [1996], Günlük [1994, 1999]) *The dimension of CS^{BI} and CS^{UN} is exactly $2|K||E_S| + |T||E_S| - |K|$.*

For Lemma 4.3 to hold it is necessary that $E_S \neq \emptyset$.

A decomposition Before investigating the polyhedral structure of the defined cut sets it will be proven that facet-defining inequalities for CS^{DI}, CS^{BI} or CS^{UN} can be facet-defining for the corresponding network design polyhedra.

Magnanti & Mirchandani [1993] and Magnanti et al. [1995] as well as Bienstock & Günlük [1996] do not consider cut sets, but directly show that given a node set S certain cut set inequalities are facet-defining for NDP^{BI} and NDP^{UN}. What they need to prove their results is that both subgraphs $G[S]$ and $G[V \setminus S]$ are connected. It is possible to decompose their results and proofs into two parts. The first part says that a certain cut set inequality is facet-defining for the corresponding cut set. The only additional argument needed then to see that this cut set inequality is facet-defining for the network design polyhedron is that the mentioned underlying subgraphs are connected.

The following theorem, which to the best of the authors knowledge is presented here for the first time, formalises this decomposition. The facial structures of the cut sets and network design polyhedra are closely related, which motivates a detailed analysis of CS^{DI}, CS^{BI} or CS^{UN}.

Theorem 4.4 *Given NDP^{DI} defined for $G = (V, A)$, let $S \subset V$ be chosen such that both $G[S]$ and $G[V \setminus S]$ are strongly connected and let CS^{DI} be the corresponding cut set. If the cut set inequality*

$$\sum_{a \in A_S,\, k \in K} \gamma_a^k f_a^k + \sum_{a \in A_S,\, t \in T} \beta_a^t x_a^t \geq \pi \tag{4.8}$$

is a facet-defining inequality for CS^{DI}, where $\gamma_a^k, \beta_a^t, \pi \in \mathbb{R}$ for all $a \in A_S, k \in K$ and $t \in T$, then it defines a facet of NDP^{DI} (in the space of the original variables).

Given NDP^{BI} (NDP^{UN}) defined for $G = (V, E)$, let $S \subset V$ be chosen such that both $G[S]$ and $G[V \setminus S]$ are connected and let CS^{BI} (CS^{UN}) be the corresponding cut set. If the cut set inequality

$$\sum_{e=ij \in E_S,\, k \in K} \gamma_{ij}^k f_{ij}^k + \sum_{e=ij \in E_S,\, k \in K} \gamma_{ji}^k f_{ji}^k + \sum_{e \in E_S,\, t \in T} \beta_e^t x_e^t \geq \pi$$

is a facet-defining inequality for CS^{BI} (CS^{UN}), where $\gamma_{ij}^k, \gamma_{ji}^k, \beta_e^t, \pi \in \mathbb{R}$ for all $e = ij \in E_S, k \in K$ and $t \in T$, then it defines a facet of NDP^{BI} (NDP^{UN}) (in the space of the original variables).

Proof. First consider the DIrected case. It can be assumed that (4.8) is given with

$$\gamma_{\bar{a}}^k = 0 \ \forall k \in K$$

for a chosen arc $\bar{a} \in A_S$, since we can add multiples of the balance constraints (4.1) to (4.8).

We will first show that the related face

$$F = \{\, (f, x) \in NDP^{DI} : \ (f, x) \text{ satisfies (4.8) with equality} \,\}$$

is nontrivial, i. e. it is not empty and it does not equal NDP^{DI}. Then, by contradiction, we will show that it defines a facet (approach 2 for facet proofs Wolsey [1998, chap 9.2.3]).

Let

$$F_S = \{\, (f,x) \in CS^{DI} :\ (f,x) \text{ satisfies (4.8) with equality} \,\}$$

be the facet of CS^{DI} defined by (4.8). Choose a point $\bar{p} = (\bar{f}, \bar{x}) \in F_S$. From $\bar{p}$ we want to construct a point $\hat{p} = (\hat{f}, \hat{x}) \in F$. Define $\hat{p}$ the following way:

$$\hat{x}^t_a := \begin{cases} \bar{x}^t_a & a \in A_S \\ M & \text{else} \end{cases} \quad \forall t \in T, \qquad \hat{f}^k_a := \bar{f}^k_a \ \forall a \in A_S, k \in K$$

where M is a large number. It remains to define $\hat{f}^k_a$ for arcs a in $A \backslash A_S = A[S] \cup A[V \backslash S]$. For $k \in K$ temporarily define the following demand vector:

$$\widetilde{d}^k_i = \begin{cases} d^k_i & \delta(i) \cap A_S = \emptyset \\ d^k_i + \bar{f}^k(\delta^-(i) \cap A^-_S) - \bar{f}^k(\delta^+(i) \cap A^+_S) & \delta(i) \cap A_S \neq \emptyset, i \in S \\ d^k_i + \bar{f}^k(\delta^-(i) \cap A^+_S) - \bar{f}^k(\delta^+(i) \cap A^-_S) & \delta(i) \cap A_S \neq \emptyset, i \in V \backslash S \end{cases}$$

Thus, if i is head or tail of an arc in the dicut A_S, we defined $\widetilde{d}_i$ to be the flow that has to leave (or enter) the node i across the cut.

It follows that $\widetilde{d}^k_S = \sum_{i \in S} \widetilde{d}^k_i = d^k_S - \bar{f}^k(A^+_S) + \bar{f}^k(A^-_S) = 0$ since $\bar{f}$ satisfies the flow conservation constraints (4.1). Similarly, $\widetilde{d}^k_{V \backslash S} = 0$. Hence a feasible flow with respect to $\widetilde{d}^k$ can be constructed that solely uses arcs in $A[S]$ and $A[V \backslash S]$. Note that the capacity is large enough. Here, again, we need $G[S]$ and $G[V \backslash S]$ to be strongly connected. Together with the flow on A_S this defines a flow $\hat{f}$ that meets all flow conservation constraints (2.2) and capacity constraints (2.4), $\hat{p} = (\hat{x}, \hat{f})$ is in NDP^{DI}. Since we did not change flow and capacity on A_S, the point still satisfies (4.8) with equality and hence $\hat{p}$ is on the face F and thus F is not empty.

Since F_S is a facet of CS^{DI}, there is a point in CS^{DI} not in F_S. From that point we construct a feasible point of NDP^{DI} using the same construction. This point then cannot be in F. It follows that $F \neq NDP^{DI}$.

We have already shown that $\emptyset \neq F \neq NDP^{DI}$. We still have to show that F is inclusion-wise maximal. We do this by contradiction. Suppose that F is not a facet. There is a face $\widetilde{F}$ of NDP^{DI} with $F \subset \widetilde{F} \neq NDP^{DI}$, where $\widetilde{F}$ is defined by

$$\sum_{a \in A,\ k \in K} \widetilde{\gamma}^k_a f^k_a + \sum_{a \in A,\ t \in T} \widetilde{\beta}^t_a x^t_a = \widetilde{\pi}. \tag{4.9}$$

Thus, every point in F satisfies (4.9). We will show that (4.9) is (4.8) up to a linear combination of flow conservation constraints (2.2) which proves that $\widetilde{F}$ induces the same face, contradicting $F \subset \widetilde{F}$.

For all $a \notin A_S$ and every $t \in T$ we can modify $\hat{p}$ by increasing the capacity $\hat{x}^t_a$. This way we obtain new points on the face F and hence

$$\widetilde{\beta}^t_a = 0 \ \forall a \notin A_S, t \in T.$$

Let $U \subseteq A[S]$ be a spanning arborescence in $G[S]$ with root r, where r is a node in S. The arborescence U exists since $G[S]$ is strongly connected. For every node $i \in S \backslash \{r\}$ there exists a unique directed path in U from r to i.

By adding a linear combination of the flow conservation constraints (2.2) to (4.9) we can assume that

$$\widetilde{\gamma}_a^k = 0 \;\; \forall a \in U, k \in K$$

That this is possible follows from the fact, that for every $k \in K$, the $S \times U$ incidence matrix defined by (the left hand side of) all flow conservation constraints (2.2) for $i \in S$ and all arcs in U has a rank of exactly $|S| - 1 = |U|$.

Now let $a_0 = uv$ be an arc in $A[S] \backslash U$ and

$$(v = i_1, i_2, ..., i_k = r)$$

a directed path in $G[S]$ from v to r with $k \geq 1$ ($v = r$ if $k = 1$), which exists since $G[S]$ is strongly connected.

We want to conclude that $\widetilde{\gamma}_{(u,v)}^k = \widetilde{\gamma}_{(i_1,i_2)}^k = \widetilde{\gamma}_{(i_2,i_3)}^k = ... = \widetilde{\gamma}_{(i_{k-1},i_k)}^k = 0$. There is a circuit in $G = (V, A)$ defined by the unique path from r to i_{k-1} in U and the arc (i_{k-1}, r). For every commodity k we can modify $\hat{p}$ by sending a circulation flow through that circuit. This way we get a new point on the face that satisfies (4.9). It follows that $\widetilde{\gamma}_{(i_{k-1},r)}^k = 0$ since $\widetilde{\gamma}_a^k = 0 \;\; \forall a \in U$. Similarly, there is a closed directed path defined by the unique path from r to i_{k-2} in U and the arcs (i_{k-2}, i_{k-1}) and (i_{k-1}, r). Again, sending a circulation flow on that path gives $\widetilde{\gamma}_{(i_{k-2},i_{k-1})}^k = 0$. We proceed inductively and get the desired result. Since $a_0 = uv$ was chosen arbitrarily $\widetilde{\gamma}_a^k = 0 \;\; \forall a \in A[S], k \in K$. The same procedure applied to $A[V \backslash S]$ gives

$$\widetilde{\gamma}_a^k = 0 \;\; \forall a \in A \backslash A_S, k \in K.$$

Now we can concentrate on coefficients of variables in the dicut. Given the above chosen arc $\bar{a}$, we first add a linear combination of flow conservation constraints (2.2) to (4.9) such that

$$\widetilde{\gamma}_{\bar{a}}^k = 0 = \gamma_{\bar{a}}^k \;\; \forall k \in K.$$

The left hand side of (4.9) now has at most $|K||A_S| + |T||A_S| - |K|$ nonzero coefficients. From Lemma 4.1 follows that F_S contains $|K||A_S| + |T||A_S| - |K|$ affinely independent points. From any of them we can construct a feasible point in F maintaining the affine independence as it has been shown for $\bar{p}$ and $\hat{p}$ above. All those points satisfy (4.9) since $F \subset \widetilde{F}$. Hence (γ, β) is the unique solution to the corresponding linear system (up to a scalar multiple) and (4.9) defines the same face as (4.8). We have shown that $F = \widetilde{F}$. This is a contradiction to $F \subset \widetilde{F}$. It follows that F is inclusion-wise maximal and together with $\emptyset \neq F \neq NDP^{DI}$ we have proven that F defines a facet of NDP^{DI}.

The proof for the BIdirected and UNdirected case is analogous. The first part of it, showing that all coefficients of (4.9) corresponding to variables not in the cut are zero, can in fact be found in Bienstock & Günlük [1996, proof of Theorem 2.2]. The important difference to the proof above is that we choose a spanning tree U in $G[S]$, which exists since $G[S]$ is connected, and that we send circulation flows (in both directions) on undirected circuits defined by edges of U and a single edge in $E[S] \backslash U$. The arguments of the second part of the proof are identical to the arguments above. ∎

Summary We have defined the cut sets, which will be investigated in this chapter and stated necessary additional definitions and assumptions. It was shown that for CS^{UN} we can assume $K^- = \emptyset$ in the following.

Moreover, we have proven that facet-defining inequalities for cut sets are facet-defining for network design polyhedra if the subgraphs $G[S]$ and $G[V \setminus S]$ are connected (undirected supply) graphs or strongly connected (directed supply graphs). With this result in mind, we will from now on concentrate on the facial structures of cut sets and develop facet-defining *MIR*-inequalities.

4.2 The cut set for single facility problems

In this section the cut sets CS^{DI}, CS^{BI} and CS^{UN} are given with a single-facility, having a capacity of $c \in \mathbb{Z}_+ \setminus \{0\}$. It follows that $|T| = 1$ and rewriting the capacity constraints (4.2), (4.5) and (4.6) with respect to the three capacity models yields:

$$\text{DIrected:} \quad \sum_{k \in K} f_a^k \leq c x_a \quad \forall a \in A_S \tag{4.10}$$

$$\text{BIdirected:} \quad \sum_{k \in K} f_{ij}^k \leq c x_e, \quad \sum_{k \in K} f_{ji}^k \leq c x_e \quad \forall e = ij \in E_S. \tag{4.11}$$

$$\text{UNdirected:} \quad \sum_{k \in K} (f_{ij}^k + f_{ji}^k) \leq c x_e \quad \forall e = ij \in E_S \tag{4.12}$$

We will state strong valid inequalities for all three sets elaborating the differences.

4.2.1 DIrected capacity constraints

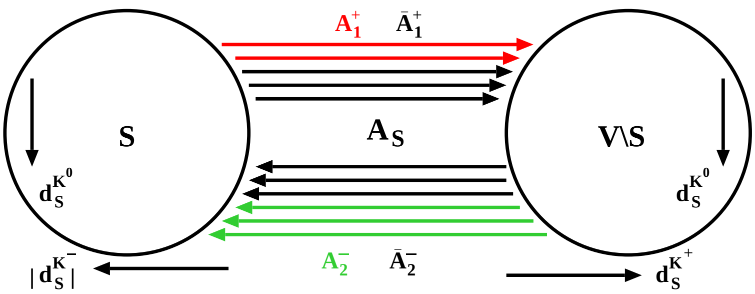


Figure 4.3: Directed cut A_S with selected arc sets A_1^+ and A_2^-

We start with an example.

Example 4.5 *Consider a single-facility, single-commodity cut set with two outflow arcs* $A_S^+ = \{a_1, a_2\}$ *and two inflow arcs* $A_S^- = \{a_3, a_4\}$*. We have to satisfy a demand of* $d_S = 7$ *and we are allowed to install capacity in units of* $c = 3$*:*

$$CS^{DI} = \text{conv}\{x \in \mathbb{Z}^4, f \in \mathbb{R}^4 \mid f_1 + f_2 - f_3 - f_4 = 7, \quad 0 \leq f_i \leq 3x_i \; \forall i \in \{1,2,3,4\}\}$$

If $x_1 = 0$ ($x_1 = 1$, $x_1 = 2$), there has to be a flow of $f_2 \geq 7$ ($f_2 \geq 4$, $f_2 \geq 1$) on arc a_2 since f_1 cannot exceed a value of 0 (3, 6). On the other hand, if $x_1 \geq 3$, then all flow can be routed through arc a_1. We can formulate the valid inequality

$$f_2 + x_1 \geq 3,$$

which is a simple flow cut inequality and defines a facet of CS^{DI}. The same inequality can be obtained by considering the valid inequality $f_2 + 3x_1 \geq 7$ and applying MIR *(Corollary 3.10).*

In the following, we will generalise the last example and develop a class of flow cut inequalities using the *MIR*-procedure defined in Section 3.4. Let Q be a subset of the commodities K. Define $f^Q(A_S) := \sum_{k \in Q} f^k(A_S)$ and similar $d_S^Q := \sum_{k \in Q} d_S^k$. If $|K| = 1$, we set $d_S := d_S^K$ and assume that $d_S > 0$ w. l. o. g..

Aggregating Summing up the flow conservation constraints (4.1) for Q gives

$$f^Q(A_S^+) - f^Q(A_S^-) = d_S^Q.$$

Let $A_1^+ \subseteq A_S^+$, $A_2^- \subseteq A_S^-$ be subsets of the arcs in the dicut A_S and $\bar{A}_1^+ := A_S^+ \setminus A_1^+$. Adding to the aggregated flow conservation constraint the aggregated capacity constraint $cx(A_1^+) \geq f^Q(A_1^+)$ and the non-negativity constraints for $A_S \setminus A_2^-$ results in the valid base inequality:

$$f^Q(\bar{A}_1^+) + cx(A_1^+) - f^Q(A_2^-) \geq d_S^Q.$$

Substituting Let $\bar{f}_a^Q$ be the slack variable of the (relaxed) capacity constraint $f_a^Q \leq cx_a$. Substituting f_a^Q for $cx_a - \bar{f}_a^Q$ for all $a \in A_2^-$ yields

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + c\big(x(A_1^+) - x(A_2^-)\big) \geq d_S^Q. \tag{4.13}$$

Note that $\bar{f}^Q(A_2^-) = cx(A_2^-) - f^Q(A_2^-) \geq 0$.

Scaling and *MIR* The $\frac{1}{c}$-*MIR* inequality for (4.13) is

$$f^Q(\bar{A}_1^+) + cx(A_2^-) - f^Q(A_2^-) + r(d_S^Q, c)\big(x(A_1^+) - x(A_2^-)\big) \geq r(d_S^Q, c)\lceil \tfrac{d_S^Q}{c} \rceil. \tag{4.14}$$

since

$$\overline{\mathcal{G}}_{d_S^Q,c}(1) = 1, \quad \mathcal{G}_{d_S^Q,c}(c) = r(d_S^Q, c) \quad \text{and} \quad \mathcal{G}_{d_S^Q,c}(-c) = -r(d_S^Q, c).$$

The inequalities (4.14) will be called **flow cut inequalities**. They were first introduced by Chopra et al. [1998] and studied in detail by Atamtürk [2002].

Recall that

$$r(d_S^Q, c) < c \iff \langle \tfrac{d_S^Q}{c} \rangle > 0 \quad \text{and} \quad r(d_S^Q, c) = c \iff \langle \tfrac{d_S^Q}{c} \rangle = 0 \iff r(d_S^Q, c)\lceil \tfrac{d_S^Q}{c} \rceil = d_S^Q.$$

Proposition 4.6 *The flow cut inequality* (4.14) *is valid for CS^{DI}.*

Proof. This follows by construction and Corollary 3.8. ∎

We will distinguish some special cases of flow cut inequalities:

Definition 4.7

- *A **simple flow cut inequality** is a flow cut inequality* (4.14) *with only outflow, that is* $A_2^- = \emptyset$*:*

$$f^Q(\bar{A}_1^+) + r(d_S^Q, c)x(A_1^+) \geq r(d_S^Q, c)\lceil \tfrac{d_S^Q}{c} \rceil$$

- *A **cut inequality** is a simple flow cut inequality with* $A_1^+ = A_S^+$*:*

$$x(A_S^+) \geq \lceil \tfrac{d_S^Q}{c} \rceil$$

In the following, necessary and sufficient conditions will be provided for (4.14) being facet-defining for CS^{DI}.

Let $\eta^Q := \lceil \frac{d_S^Q}{c} \rceil$ and $r^Q := r(d_S^Q, c)$ and remember that $K^+ \neq \emptyset$. The following Lemma states necessary conditions for flow cut inequalities (4.14) to be facet-defining.

Necessary conditions A valid inequality for CS^{DI} is called trivial if it is equivalent to a non-negativity constraint (4.3) or a capacity constraint (4.10) up to a linear combination of flow conservation constraints (4.1).

Lemma 4.8 *Let* $A_1^+ \subseteq A_S^+$, $A_2^- \subseteq A_S^-$, $Q \subseteq K$, $d_S^Q \geq 0$*. If* (4.14) *is a nontrivial facet-defining inequality for* CS^{DI}*, then every of the following statements is true:*

i) $r^Q < c$ *and* $A_1^+ \neq \emptyset$.

ii) If (4.14) *is a simple flow cut inequality with* $A_1^+ \neq A_S^+$ *and* $Q \subseteq K^+$*, then* $|Q| = 1$ *or* $d_S^Q > c$.

iii) If (4.14) *is a cut inequality, then* $\eta^Q = \eta^{K^+}$.

iv) If (4.14) *is a cut inequality and* $|A_S^+| > 1$*, then* $d_S^{K^+} > c$ *or* $A_S^- \neq \emptyset$.

Proof. i) If $r^Q = c$, then inequality (4.14) reduces to $f^Q(\bar{A}_1^+) + cx(A_1^+) - f(A_2^-) \geq d_S^Q$ which is the sum of $f^Q(A_S^+) - f^Q(A_S^-) \geq d_S^Q$, non-negativity constraints for $A_S^- \backslash A_2^-$ and capacity constraints for A_1^+. Hence it is not a facet or trivial.

Else if $A_1^+ = \emptyset$, then inequality (4.14) can be written as

$$f^Q(A_S^+) - f^Q(A_2^-) + (c - r^Q)x(A_2^-) \geq r^Q\eta^Q = d_S^Q - (\eta^Q - 1)(c - r^Q)$$

(see Lemma 3.11 i)), which is dominated by $f^Q(A_S^+) - f^Q(A_2^-) \geq d_S^Q$ since $\eta^Q \geq 1$ and $c > r^Q$.

ii) Suppose $d_S^Q \leq c$ and $Q = \{q_1, ..., q_l\}$ with $l \geq 2$. It follows $d_S^{q_i} \leq c \ \ \forall i \in \{1, .., l\}$, $d_S^Q = r^Q = \sum_{i=1}^l d_S^{q_i} = \sum_{i=1}^l r^{q_i}$ and $\eta^Q = \eta^{q_i} = 1$. So (4.14) is the sum of the l valid simple flow cut inequalities (different from flow conservation constraints):

$$f^{q_i}(\bar{A}_1^+) + r^{q_i}x(A_1^+) \geq r^{q_i}.$$

They differ if $A_1^+ \neq A_S^+$, which is equivalent to $\bar{A}_1^+ \neq \emptyset$.

iii) By definition of K^+, $d_S^Q \le d_S^{K^+}$ and thus $\lceil \frac{d_S^Q}{c} \rceil \le \lceil \frac{d_S^{K^+}}{c} \rceil$. If $\lceil \frac{d_S^Q}{c} \rceil < \lceil \frac{d_S^{K^+}}{c} \rceil$, then $x(A_S^+) \ge \lceil \frac{d_S^{K^+}}{c} \rceil$ dominates $x(A_S^+) \ge \lceil \frac{d_S^Q}{c} \rceil$.

iv) Suppose $|A_S^+| \ge 2$, $d_S^Q < c$ and $A_S^- = \emptyset$. Choose $B^+ \subset A_S^+$ and set $\bar{B}^+ := A_S^+ \backslash B^+ \neq \emptyset$. We can formulate two valid simple flow cut inequalities:

$$f^Q(\bar{B}^+) + r^Q x(B^+) \ge r^Q \eta^Q \qquad \text{and} \qquad f^Q(B^+) + r^Q x(\bar{B}^+) \ge r^Q \eta^Q$$

Adding them up results in $f^Q(A_S^+) + d_S^Q x(A_S^+) \ge 2d_S^Q$. Note that $r^Q = d_S^Q$ and $\eta^Q = 1$. But from $A_S^- = \emptyset$ follows that $f^Q(A_S^+) = d_S^Q$ and hence the cut inequality

$$x(A_S^+) \ge 1$$

is the sum of valid inequalities (different from flow conservation constraints). ■

The cut inequality After stating the necessary conditions for the flow cut inequalities (4.14) to be facet-defining for CS^{DI}, sufficient conditions will now be provided. Cut inequalities are crucial both from the theoretical and the computational point of of view. Because of that, we handle them here separately. Moreover, the techniques needed to prove the following result form the basis for each of the facet theorems of this chapter.

Theorem 4.9 *The cut inequality $x(A_S^+) \ge \eta^{K^+}$ is facet-defining for CS^{DI} if and only if $r^{K^+} < c$ and one of the following conditions holds:*

i) $|A_S^+| = 1$

ii) $A_S^- \neq \emptyset$

iii) $d_S^{K^+} > c$

Proof. Necessity: see Lemma 4.8.

Sufficiency: Set $\eta^{K^+} := \lceil \frac{d_S^{K^+}}{c} \rceil$. Remember that by Lemma 3.11

$$c\eta^{K^+} = d_S^{K^+} + c - r^{K^+}.$$

and that $r^{K^+} < c$ is equivalent to $\frac{d_S^{K^+}}{c} \notin \mathbb{Z}$. We will show that the related face

$$F_{DI} = \{\, (f, x) \in CS^{DI} : \; x(A_S^+) = \eta^{K^+} \,\}$$

is nontrivial i. e., it is not empty and it does not equal CS^{DI}. Then by contradiction, we will show that it defines facets (approach 2 for facet proofs Wolsey [1998, chap 9.2.3]).

Choose $a_0 \in A_S^+$. Now construct a feasible point $p_0 = (\bar{f}, \bar{x})$ on the face F_{DI} the following way: Set $\bar{x}_{a_0} = \eta^{K^+}$ and satisfy the demand for commodities $k \in K^+$ by sending a flow of $\bar{f}_{a_0}^k = d_S^k$ on a_0.

Hence p_0 fulfils the flow conservation constraints with respect to K^+. It meets the capacity constraints for a_0 since

$$\sum_{k \in K} \bar{f}_{a_0}^k = \sum_{k \in K^+} d_S^k = d_S^{K^+} < d_S^{K^+} + c - r^{K^+} = c\eta^{K^+} = c\bar{x}_{a_0}.$$

It remains to route the demands for K^- (flow for K^0 is fixed to zero). If $K^- \neq \emptyset$ then $A_S^- \neq \emptyset$ (see Section 4.1). Additionally, choose an arc $\bar{a}_0 \in A_S^-$ and install a large integer capacity: $\bar{x}_{\bar{a}_0} = M$.

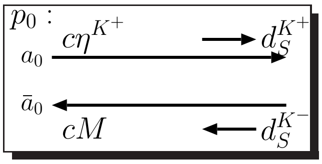


Figure 4.4: a_0 and $\bar{a}_0$ are used to route the total flow.

Satisfy all demand for $k \in K^-$ by sending a flow of $\bar{f}^k_{\bar{a}_0} = d^k_S$. If M is large enough, then p_0 is on the face F_{DI} since all demands and capacity constraints are satisfied and $\bar{x}(A^+_S) = \eta^{K^+}$.

We have shown that F_{DI} is not empty. Modifying p_0 by setting $\bar{x}_{a_0} = \eta^{K^+} + 1$ gives a point that is in CS^{DI} but not on the face F_{DI}. Hence $\emptyset \neq F_{DI} \neq CS^{DI}$. F_{DI} is nontrivial.

We will now prove that F_{DI} is inclusion-wise maximal by contradiction. Suppose F_{DI} is not a facet. There is a face F of CS^{DI} with $F_{DI} \subset F \neq CS^{DI}$. Let F be defined by

$$\beta^T x + \gamma^T f = \pi, \tag{4.15}$$

where β, γ are vectors of appropriate dimension and $\pi \in \mathbb{R}$.

We will show that (4.15) is a multiple of $x(A^+_S) = \eta^{K^+}$ up to a linear combination of flow conservation constraints contradicting $F_{DI} \subset F$.

Since multiples of the $|K|$ flow conservation constraints may be added to (4.15) without changing the induced face, $\gamma^k_{a_0} = 0 \;\; \forall k \in K$ can be assumed. Note that so far we have not used any of the conditions $i), ii), iii)$. However, we now distinguish several cases.

If $|A^+_S| = 1$ and $A^-_S = \emptyset$ we have finished because (4.15) reduces to $\beta_{a_0} x_{a_0} = \beta_{a_0} \eta^{K^+}$.

Assume that $A^-_S \neq \emptyset$. We can modify the point p_0 by setting $\bar{x}_{\bar{a}_0} = M + 1$. This gives a new point on the face and since $\bar{a}_0$ was arbitrary

$$\beta_a = 0 \;\; \forall a \in A^-_S.$$

Modifying p_0 by simultaneously increasing flow on a_0 and $\bar{a}_0$ by a small amount for every commodity $k \in K$ respectively changes neither a flow conservation nor a capacity constraint and hence

$$\gamma^k_a = 0 \;\; \forall k \in K, a \in A^-_S.$$

The proof is complete for $|A^+_S| = 1$ and $A^-_S \neq \emptyset$ because (4.15) again reduces to $\beta_{a_0} x_{a_0} = \beta_{a_0} \eta^{K^+}$. We can assume that $|A^+_S| > 1$ in the following.

Choose $a \in A^+_S$ different from a_0 and construct a point p from p_0 by setting $\bar{x}_{a_0} = \eta^{K^+} - 1$ and $\bar{x}_a = 1$. The maximum amount of flow that can be routed on a_0 is now $c(\eta^{K^+} - 1) = d^{K^+}_S - r^{K^+}$. Note that $\eta^{K^+} \geq 1$ and $d^{K^+}_S \geq r^{K^+}$.

It depends on $d^{K^+}_S$ how to reroute the flow. Assume first that $d^{K^+}_S < c$. It follows that $d^{K^+}_S = r^{K^+}$ and the capacity on a_0 equals zero. We simply copy the flow from a_0 to a. Hence p is defined as p_0 but for arc a instead of a_0.

If $d^{K^+}_S > c$ then $d^{K^+}_S > r^{K^+}$ and $(\eta^{K^+} - 1) \geq 1$. We construct the point p the following way. There is still capacity on a_0. Reroute a total flow of exactly r^{K^+} such that the flow is positive on both arcs a_0 and a for every positive commodity. We have to change the flow of every positive commodity.

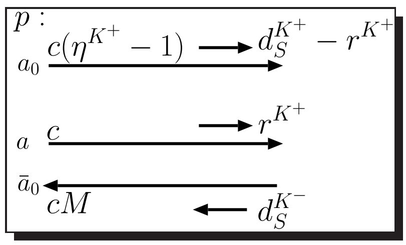


Figure 4.5: a is used to reroute the flow.

In both cases the capacity on a is not saturated and the new point p is on the face F_{DI} because we did not change the total capacity on A^+_S. Flow conservation and capacity constraints are still satisfied.

The proof is complete if we can show that

$$\gamma_a^k = 0 \;\; \forall k \in K, a \in A_S^+$$

because plugging in the points p_0, p into (4.15) gives $\beta_{a_0}\eta^{K+} = \pi$ and $\beta_{a_0}(\eta^{K+} - 1) + \beta_a = \pi$. It then follows that

$$\beta_{a_0} = \beta_a \;\; \forall a \in A_S^+$$

since a was arbitrary. Hence (4.15) reduces to $\beta_{a_0} x(A_S^+) = \beta_{a_0}\eta^{K+}$.

First suppose that $A_S^- \neq \emptyset$. We can modify p by increasing flow on a and $\bar{a}_0$. This results in

$$\gamma_a^k = 0 \;\; \forall k \in K, a \in A_S^+.$$

Finally assume that $A_S^- = \emptyset$ and $d_S^{K+} > c$. In this case $K = K^+$ (see Section 4.1). We can modify p by decreasing flow for $k \in K^+$ on a_0 and increasing it on a. This gives

$$\gamma_a^k = 0 \;\; \forall k \in K^+ = K, a \in A_S^+.$$

The last perturbation of the flow for p was only possible because $d_S^{K+} > c$ and we could construct the point p as mentioned.

We have shown that (4.15) is a multiple of $x(A_S^+) \geq \eta^{K+}$ plus a linear combination of flow conservation constraints. Thus F and F_{DI} induce the same face which contradicts $F_{DI} \subset F$. It follows that F_{DI} is inclusion-wise maximal and together with $\emptyset \neq F_{DI} \neq CS^{DI}$ it defines a facet of CS^{DI}. The proof is complete. ∎

General flow cut inequalities It has to be mentioned that Atamtürk [2002, Theorem 2] is not correct. It says that in the single-commodity case the flow cut inequality (4.14) is facet-defining for CS^{DI} if and only if $r(d_S, c) < c$ and $A_1^+ \neq \emptyset$. From Lemma 4.8 iv) a counter-example can be constructed. In the proof of Atamtürk [2002, Theorem 2] a point is considered similar to p and it is implicitly assumed that there is still capacity on a_0 after deleting one unit. This is only true if $d_S^{K+} > c$. The last proof gives an idea of how to fix this. We state a modified version of Atamtürk [2002, Theorem 2].

Theorem 4.10 (Atamtürk [2002]) *Let* $|K| = 1$, $Q = K$. *The flow cut inequality* (4.14) *is facet-defining for* CS^{DI} *if and only if* $r(d_S, c) < c$, $A_1^+ \neq \emptyset$ *and one of the following conditions holds:*

i) (4.14) *is a cut inequality and* $|A_S^+| = 1$

ii) (4.14) *is a cut inequality,* $|A_S^+| > 1$ *and* $A_S^- \neq \emptyset$

iii) (4.14) *is a cut inequality,* $|A_S^+| > 1$ *and* $d_S^Q > c$

iv) (4.14) *is not a cut inequality (* $\iff$ $\bar{A}_1^+ \neq \emptyset$ *or* $A_2^- \neq \emptyset$*).*

Equality (4.1) *and inequalities* (4.10), (4.3) *and* (4.14) *completely describe* CS^{DI}.

The last sentence of this theorem is a crucial result in the theory of strong valid inequalities for network design polyhedra. When dropping the integer constraints for design variables it suffices to add all flow cut inequalities to the initial formulation given by (4.1), (4.10) and (4.3) to maintain a complete description of CS^{DI} in the single-commodity, single-facility case.

Example 4.5 (continued) *All nontrivial facet-defining inequalities of*

$$CS^{DI} = \text{conv}\{x \in \mathbb{Z}^4, f \in \mathbb{R}^4 \mid f_1 + f_2 - f_3 - f_4 = 7 \\ 0 \le f_i \le 3x_i \;\; \forall i \in \{1,2,3,4\}\}$$

are the following flow cut inequalities:

$$\begin{aligned}
f_2 + x_1 &\ge 3 && (A_1^+ = \{1\},\ A_2^- = \emptyset) \quad \textit{simple flow cut inequality}\\
f_2 + 2x_3 - f_3 + x_1 &\ge 3 && (A_1^+ = \{1\},\ A_2^- = \{3\})\\
f_2 + 2x_4 - f_4 + x_1 &\ge 3 && (A_1^+ = \{1\},\ A_2^- = \{4\})\\
f_2 + 2x_3 + 2x_4 - f_3 - f_4 + x_1 &\ge 3 && (A_1^+ = \{1\},\ A_2^- = \{3,4\})\\
f_1 + x_2 &\ge 3 && (A_1^+ = \{2\},\ A_2^- = \emptyset) \quad \textit{simple flow cut inequality}\\
f_1 + 2x_3 - f_3 + x_2 &\ge 3 && (A_1^+ = \{2\},\ A_2^- = \{3\})\\
f_1 + 2x_4 - f_4 + x_2 &\ge 3 && (A_1^+ = \{2\},\ A_2^- = \{4\})\\
f_1 + 2x_3 + 2x_4 - f_3 - f_4 + x_2 &\ge 3 && (A_1^+ = \{2\},\ A_2^- = \{3,4\})\\
x_1 + x_2 &\ge 3 && (A_1^+ = \{1,2\},\ A_2^- = \emptyset) \quad \textit{cut inequality}\\
2x_3 - f_3 + x_1 + x_2 &\ge 3 && (A_1^+ = \{1,2\},\ A_2^- = \{3\})\\
2x_4 - f_4 + x_1 + x_2 &\ge 3 && (A_1^+ = \{1,2\},\ A_2^- = \{4\})\\
2x_3 + 2x_4 - f_3 - f_4 + x_1 + x_2 &\ge 3 && (A_1^+ = \{1,2\},\ A_2^- = \{3,4\})
\end{aligned}$$

The following theorem generalises Theorem 4.10 to the multi-commodity case.

Theorem 4.11 (Atamtürk [2002]) *Let $Q \subseteq K^+$. The flow cut inequality* (4.14) *is facet-defining for CS^{DI} if $A_1^+, \bar{A}_1^+, A_2^-, \bar{A}_2^- \ne \emptyset$, $r(d_S^Q, c) < c$.*

Atamtürk [2002] only states the theorem without proving it. In Section 4.2.2.2 we prove a similar result (Theorem 4.23) for cut sets with undirected supply graphs. Many of the ideas developed there to handle the multi-commodity case can be used to prove the result above based on the proof of Atamtürk [2002, Theorem 2] and the proof of Theorem 4.9.

Remark 4.12 *The condition $Q \subseteq K^+$ indicates that it is promising to aggregate positive commodities only. This is the only case investigated by Atamtürk [2002]. Due to the symmetry of the cut sets of S and $V \backslash S$, there is an analogue to Theorem 4.11 when $Q \subseteq K^-$.*

However, note that there is no reason to drop the case that Q contains both positive and negative commodities. In fact, it is an open question if the flow cut inequalities (4.14) *are strong in that case. Switching between the cut set for S and the cut set for $V \backslash S$ is equivalent to multiplying every flow conservation constraint by -1. A positive commodity with respect to S is a negative commodity with respect to $V \backslash S$. In the following we show that the flow cut inequality* (4.14) *with $d_S^Q < 0$ induces the same face as the flow cut inequality obtained after switching to $V \backslash S$ and considering the same commodity subset Q. Suppose that $d_S^Q < 0$ and assume that $r(d_S^Q, c) < c$. It follows that $\lceil \frac{d_S^Q}{c} \rceil = -\lfloor \frac{|d_S^Q|}{c} \rfloor$ and in Lemma 3.11 we showed that $r(d_S^Q, c) = c - r(|d_S^Q|, c)$.*

Set $r := r(|d_S^Q|, c)$. We can write (4.14) *as:*

$$\begin{aligned}
&f^Q(\bar{A}_1^+) + cx(A_2^-) - f^Q(A_2^-) + \big(c - r\big)\big(x(A_1^+) - x(A_2^-)\big)\\
&\qquad \ge -\big(c - r\big)\lfloor \tfrac{|d_S^Q|}{c} \rfloor = r\lceil \tfrac{|d_S^Q|}{c} \rceil - r - c\lfloor \tfrac{|d_S^Q|}{c} \rfloor = r\lceil \tfrac{|d_S^Q|}{c} \rceil - r - c\lceil \tfrac{|d_S^Q|}{c} \rceil + c.
\end{aligned}$$

Using $c\lceil\frac{|d_S^Q|}{c}\rceil = |d_S^Q| + c - r$ *(see Lemma 3.11) and adding the flow conservation constraint* $-d_S^Q = |d_S^Q| = f^Q(A_S^-) - f^Q(A_S^+)$ *gives*

$$f^Q(\bar{A}_2^-) + cx(A_1^+) - f^Q(A_1^+) + r\big(x(A_2^-) - x(A_1^+)\big) \geq r\lceil\tfrac{|d_S^Q|}{c}\rceil,$$

which is inequality (4.14) *but for the cut set of* $V\backslash S$*. Hence both inequalities induce the same face of* CS^{DI}.

Since the cut set of S and $V\backslash S$ are identical, this leads to the following corollary of Theorem 4.11:

Corollary 4.13 *Let* $Q^- \subseteq K^-$*. The flow cut inequality*

$$f^{Q^-}(\bar{A}_2^-) + cx(A_1^+) - f^{Q^-}(A_1^+) + r\big(x(A_2^-) - x(A_1^+)\big) \geq r\lceil\tfrac{|d_S^{Q^-}|}{c}\rceil, \qquad (4.16)$$

where $r := r(|d_S^{Q^-}|, c)$*, is facet-defining for* CS^{DI} *if* A_1^+, $\bar{A}_1^+$, A_2^-,$\bar{A}_2^- \neq \emptyset$ *and* $r(|d_S^{Q^-}|, c) < c$.

Summary In this section we have considered the cut set with DIrected capacity constraints in the single-facility case, as investigated in Atamtürk [2002]. We have stated necessary and sufficient conditions for a large class of cut set inequalities called flow cut inequalities to be facet-defining for CS^{DI}. In the single-commodity, single-facility case those inequalities suffice to completely describe the cut set. Facet-defining for the relaxation CS^{DI} with respect to S are facet-defining for NDP^{DI} if both $G[S]$ and $G[V\backslash S]$ are strongly connected as already shown in Theorem 4.4.

In the remainder of this chapter similar results for the cut sets CS^{BI} and CS^{UN} for undirected supply graphs will be proven.

4.2.2 BIdirected and UNdirected capacity constraints

Given a node set $S \subset V$ of the undirected supply graph $G = (V, E)$ and a single-facility we will consider the cut $E_S := \delta(S) \neq \emptyset$ and the cut sets CS^{BI} and CS^{UN} in this section.

After stating the necessary definitions, we will transform the results of the last section to undirected supply graphs.

A class of flow cut inequalities analogous to (4.14) will be introduced, generalising inequalities of Magnanti & Mirchandani [1993] and Bienstock & Günlük [1996]. We will give conditions for those flow cut inequalities to be facet-defining for CS^{BI} and CS^{UN}.

It turns out that these conditions depend on the capacity model. Moreover, the introduced flow cut inequalities do not suffice to completely describe the BIdirected and UNdirected cut sets in the single-commodity, single-facility case which is reflected by a new class of facet-defining cut set inequalities that has no analogue for CS^{DI}.

4.2.2.1 Cut set inequalities and necessary conditions

Now we will develop strong valid inequalities for CS^{BI} and CS^{BI}, transforming results of the last section. To obtain a similar inequality to (4.14), we simply apply the same *MIR* procedure.

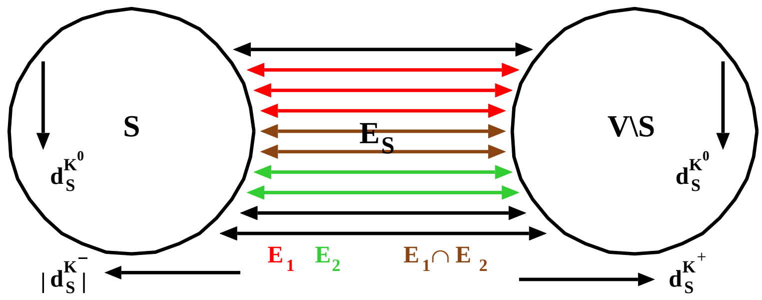


Figure 4.6: Undirected cut E_S with selected edge sets E_1 and E_2 and demand directions

Given a subset Q of K let

$$f^Q(E_S^+) := \sum_{k \in Q} f^k(E_S^+) \quad \text{and} \quad f^Q(E_S^-) := \sum_{k \in Q} f^k(E_S^-).$$

Set $d_S^Q := \sum_{k \in Q} d_S^k$, $\eta^Q := \lceil \frac{d_S^Q}{c} \rceil$ and $r^Q := r(d_S^Q, c)$. If $|K| = 1$, we set $d_S := d_S^k$ and assume that $d_S > 0$ w. l. o. g., since the cut sets of S and $V \backslash S$ are identical.

Aggregating Summing up the flow conservation constraints for Q results in

$$f^Q(E_S^+) - f^Q(E_S^-) = d_S^Q.$$

Let E_1, E_2 be two subsets of the cut E_S and $\bar{E}_1 := E_S \backslash E_1$. Adding to the aggregated flow conservation constraint the aggregated (BIdirected) capacity constraint $cx(E_1) \geq f^Q(E_1^+)$ and the non-negativity constraints for $E_S \backslash E_2$ gives the base inequality

$$f^Q(\bar{E}_1^+) + cx(E_1) - f^Q(E_2^-) \geq d_S^Q$$

valid for CS^{BI} and CS^{UN}. Note that BIdirected capacity constraints are valid for CS^{UN}.

Substituting Let $\bar{f}_{ji}^Q$ be the slack variable of the (relaxed BIdirected) capacity constraint $f_{ji}^Q \leq cx_e$. Substituting f_{ji}^Q for $cx_e - \bar{f}_{ji}^Q$ for all $e = ij \in E_2$ gives

$$f^Q(\bar{E}_1^+) + \bar{f}^Q(E_2^-) + c\big(x(E_1) - x(E_2)\big) \geq d_S^Q. \tag{4.17}$$

Note that $\bar{f}^Q(E_2^-) = cx(E_2) - f^Q(E_2^-) \geq 0$.

Scaling and *MIR* Apply *MIR* (Corollary 3.8). The $\frac{1}{c}$-*MIR* inequality for (4.17) is

$$f^Q(\bar{E}_1^+) + cx(E_2) - f^Q(E_2^-) + r(d_S^Q, c)\big(x(E_1) - x(E_2)\big) \geq r(d_S^Q, c)\lceil \tfrac{d_S^Q}{c} \rceil. \tag{4.18}$$

The inequalities (4.18) will be called **flow cut inequalities** for CS^{BI} and CS^{UN} and can be seen as the undirected analogue to (4.14). They are presented here in this general form for the first time.

In contrast to inequality (4.14), both E_1 and E_2 are chosen from the same set. Hence $E_1 \cap E_2 \neq \emptyset$ is possible. It is going to be investigated how to choose E_1 and E_2 such that (4.18) is strong.

Proposition 4.14 *The flow cut inequality* (4.18) *is valid for* CS^{UN} *and* CS^{BI}.

Proof. This follows by construction and from Lemma 4.2 and Corollary 3.8. ■

Definition 4.15 *As for the DIrected capacity model we call flow cut inequalities **simple** if there is no inflow,* $E_2 = \emptyset$*:*

$$f^Q(\bar{E}_1) + r^Q x(E_1) \geq r^Q \eta^Q$$

*A **cut inequality** is a simple flow cut inequality with* $E_1 = E_S$*:*

$$x(E_S) \geq \eta^Q$$

The demand d_S^Q might be negative for CS^{BI} but with the same arguments as in Remark 4.12 we can assume that $d_S^Q \geq 0$ w. l. o. g., because if $d_S^Q < 0$ we can switch to the cut sets of $V \backslash S$ and get

$$f^Q(\bar{E}_2^-) + cx(E_2) - f^Q(E_1^+) + r(-d_S^Q, c)\big(x(E_1) - x(E_2)\big) \geq r(-d_S^Q, c)\lceil \tfrac{-d_S^Q}{c} \rceil \tag{4.19}$$

as a valid inequality for CS^{BI}. Note again that $d_{V\backslash S}^Q = -d_S^Q$.

Example 4.16 *Similar to Example 4.5 we define cut sets with two edges,* $d_S = 7$ *and* $c = 3$*:*

$$\begin{aligned} CS^{BI} = \text{conv}\{x \in \mathbb{Z}^2, f \in \mathbb{R}^4 \mid\ & f_1 + f_2 - f_3 - f_4 = 7, \\ & 0 \leq f_i \leq 3x_1\ \forall i \in \{1,3\}, \\ & 0 \leq f_i \leq 3x_2\ \forall i \in \{2,4\}\} \end{aligned}$$

$$\begin{aligned} CS^{UN} = \text{conv}\{x \in \mathbb{Z}^2, f \in \mathbb{R}^4 \mid\ & f_1 + f_2 - f_3 - f_4 = 7, \\ & f_1 + f_3 \leq 3x_1, \\ & f_2 + f_4 \leq 3x_2, \\ & 0 \leq f_i\ \forall i \in \{1, ..., 4\}\} \end{aligned}$$

Note that f_1*,* f_2 *are the forward flows on the first and second edge respectively, while* f_3 *and* f_4 *are the corresponding backward flows. With* $E_1 \neq \emptyset$ *we can formulate the following flow cut inequalities which are valid for* CS^{BI} *and* CS^{UN}*:*

$$f_2 + x_1 \geq 3 \quad (E_1 = \{1\},\ E_2 = \emptyset) \quad \textit{simple flow cut inequality} \tag{4.20a}$$
$$f_2 + 2x_1 - f_3 + x_1 \geq 3 \quad (E_1 = \{1\},\ E_2 = \{1\}) \tag{4.20b}$$
$$f_2 + 2x_2 - f_4 + x_1 \geq 3 \quad (E_1 = \{1\},\ E_2 = \{2\}) \tag{4.20c}$$
$$f_2 + 2x_1 + 2x_2 - f_3 - f_4 + x_1 \geq 3 \quad (E_1 = \{1\},\ E_2 = \{1,2\}) \tag{4.20d}$$
$$f_1 + x_2 \geq 3 \quad (E_1 = \{2\},\ E_2 = \emptyset) \quad \textit{simple flow cut inequality} \tag{4.20e}$$
$$f_1 + 2x_1 - f_3 + x_2 \geq 3 \quad (E_1 = \{2\},\ E_2 = \{1\}) \tag{4.20f}$$
$$f_1 + 2x_2 - f_4 + x_2 \geq 3 \quad (E_1 = \{2\},\ E_2 = \{2\}) \tag{4.20g}$$
$$f_1 + 2x_1 + 2x_2 - f_3 - f_4 + x_2 \geq 3 \quad (E_1 = \{2\},\ E_2 = \{1,2\}) \tag{4.20h}$$
$$x_1 + x_2 \geq 3 \quad (E_1 = \{1,2\},\ E_2 = \emptyset) \quad \textit{cut inequality} \tag{4.20i}$$
$$2x_1 - f_3 + x_1 + x_2 \geq 3 \quad (E_1 = \{1,2\},\ E_2 = \{1\}) \tag{4.20j}$$

$$2x_2 - f_4 + x_1 + x_2 \geq 3 \quad (E_1 = \{1,2\},\ E_2 = \{2\}) \tag{4.20k}$$
$$2x_1 + 2x_2 - f_3 - f_4 + x_1 + x_2 \geq 3 \quad (E_1 = \{1,2\},\ E_2 = \{1,2\}) \tag{4.20l}$$

These are the inequalities from Example 4.5 but x_3 was replaced by x_1 and x_4 by x_2. Now some of these inequalities are weak or trivial. For instance (4.20b) *can be written as $f_2 + 3x_1 - f_3 \geq 3$ which is the sum of $f_2 + f_1 - f_3 - f_4 \geq 7$, capacity- and non-negativity constraints. The same is true for* (4.20g) *and* (4.20l).

Necessary Conditions The following Lemma provides necessary conditions for flow cut inequalities to be facet-defining. A crucial observation in this context is, that for BIdirected problems it has to be $\boldsymbol{E_1 \backslash E_2 \neq \emptyset}$ and for UNdirected problems $\boldsymbol{E_1 \cap E_2 = \emptyset}$ is needed additionally.

A valid inequality for CS^{BI} (resp. CS^{UN}) is called trivial if it is equivalent to a non-negativity constraint (4.7) or a capacity constraint (4.11) (resp. (4.12)) up to a linear combination of flow conservation constraints (4.4).

Lemma 4.17 *Let $E_1, E_2 \subseteq E_S$, $Q \subseteq K$, $d_S^Q \geq 0$. If* (4.18) *is a nontrivial facet-defining inequality for CS^{BI} or CS^{UN}, then every of the following statements is true:*

i) $r^Q < c$ and $E_1 \backslash E_2 \neq \emptyset$.

ii) If (4.18) *is a simple flow cut inequality with $E_1 \neq E_S$ and $Q \subseteq K^+$, then $|Q| = 1$ or $d_S^Q > c$.*

iii) If (4.18) *is a cut inequality, then $\eta^Q = \eta^{K^+}$.*

iv) If (4.18) *is facet-defining for CS^{UN}, then $E_1 \cap E_2 = \emptyset$.*

Proof. i) If $r^Q = c$, then inequality (4.18) reduces to $f^Q(\bar{E}_1^+) + cx(E_1) - f^Q(E_2^-) \geq d_S^Q$ which is the sum of $f^Q(E_S^+) - f^Q(E_S^-) = d_S^Q$, non-negativity constraints for $E_S^- \backslash E_2^-$ and capacity constraints for E_1^+. Hence it either trivial or does not define a facet.

Assume that $r^Q < c$ and $E_1 \backslash E_2 = \emptyset$. It follows $E_1 \subseteq E_2$. Inequality (4.18) can be written as

$$f^Q(\bar{E}_1^+) + cx(E_1) - f^Q(E_2^-) + (c - r^Q)x(E_2 \backslash E_1) \geq r^Q \eta^Q = d_S^Q - (\eta^Q - 1)(c - r^Q)$$

(see Lemma 3.11 i)), which is dominated by

$$f^Q(E_S^+) - f^Q(E_2^-) \geq d_S^Q$$

since $\eta^Q \geq 1$ and $c > r^Q$. Note that $cx(E_1) \geq f^Q(E_1^+)$ and $(c - r^Q)x(E_2 \backslash E_1) \geq 0$.

ii) see Lemma 4.8

iii) see Lemma 4.8

iv) If $E_1 \cap E_2 \neq \emptyset$, inequality (4.18) can be written as

$$f^Q(\bar{E}_1^+) + cx(E_2 \backslash E_1) + \boldsymbol{cx(E_1 \cap E_2)} - f^Q(E_2^- \backslash E_1^-) - \boldsymbol{f^Q(E_1^- \cap E_2^-)}$$
$$+ r(d_S^Q, c)\big(x(E_1 \backslash E_2) - x(E_2 \backslash E_1)\big) \geq r(d_S^Q, c)\lceil \tfrac{d_S^Q}{c} \rceil.$$

But formulating the flow cut inequality (4.18) with $E_1^* := E_1 \backslash E_2$ and $E_2^* := E_2 \backslash E_1$, implying $\bar{E}_1^* = \bar{E}_1 \cup (E_1 \cap E_2)$, results in:

$$f^Q(\bar{E}_1^+) + \boldsymbol{f^Q(E_1^+ \cap E_2^+)} + cx(E_2 \backslash E_1) - f^Q(E_2^- \backslash E_1^-) \\ + r(d_S^Q, c)\big(x(E_1 \backslash E_2) - x(E_2 \backslash E_1)\big) \geq r(d_S^Q, c)\lceil \tfrac{d_S^Q}{c} \rceil.$$

Since $cx(E_1 \cap E_2) - f^Q(E_1^+ \cap E_2^+) - f^Q(E_1^- \cap E_2^-) \geq 0$ is valid for CS^{UN}, (4.18) with $E_1 \cap E_2 \neq \emptyset$ is the sum of valid inequalities (different from flow conservation constraints). ∎

Remark 4.18 *General flow cut inequalities of type* (4.18) *(*(4.19)*) can be facet-defining if Q is a subset of K^+ (K^-), as we will see. For CS^{BI} we will not consider the more general case, that Q contains both positive and negative commodities. There is no reason to drop that case when separating flow cut inequalities. It is unknown if they are strong or even facet-defining when Q contains positive and negative commodities.*

Bienstock & Günlük [1996] restrict themselves to $Q \subseteq K^+$ and do not consider the more general case $Q \subseteq K$ neither, just as Atamtürk [2002] (see Theorem 4.11).

Remember that for CS^{UN} we assume $K^- = \emptyset$.

A new class of cut set inequalities It follows from Theorem 4.10 (Atamtürk [2002]) that a complete linear description of CS^{DI} can be derived by adding all flow cut inequalities of type (4.14) to the initial formulation when $|K| = 1$. This is not true for CS^{BI} (CS^{UN}) and the flow cut inequalities of type (4.18).

Example 4.16 (continued) *When adding all flow cut inequalities* (4.20a),...,(4.20l) *to the LP-relaxations of CS^{DI} and CS^{UN}, both resulting polyhedra still have the two fractional vertices $(\frac{1}{2}, \frac{15}{2}, 1, 0, \frac{1}{2}, \frac{5}{2})$ and $(\frac{15}{2}, \frac{1}{2}, 0, 1, \frac{5}{2}, \frac{1}{2})$. But we can formulate two valid cut set inequalities cutting off these points, namely:*

$$3x_1 + 2x_2 + f_3 - f_1 \geq 2$$

and

$$3x_2 + 2x_1 + f_4 - f_2 \geq 2.$$

The surprising result now is, that these two inequalities together with all flow cut inequalities describe all non-trivial facets of CS^{BI} and CS^{UN}, which yields a complete description of both polyhedra.

The inequalities given in the last example belong to a new class of valid cut set inequalities introduced by the following theorem.

Theorem 4.19 *Let E_1 be a subset of the cut E_S and let Q be a subset of the commodities K with $d_S^Q \geq 0$. Set $r^Q := r(d_S^Q, c)$. The following inequality is valid for CS^{BI} and CS^{UN}:*

$$cx(E_1) + \big(c - r^Q\big)x(\bar{E}_1) + f^Q(E_1^-) - f^Q(E_1^+) \geq c - r^Q. \tag{4.21}$$

Proof. If $r^Q = c$ then inequality (4.21) reduces to $cx(E_1) - f^Q(E_1^+) + f^Q(E_1^-) \geq 0$, which is valid because of the capacity constraint $cx(E_1) \geq f^Q(E_1^+)$. We can suppose that $r^Q < c$.

First assume that $x(\bar{E}_1) = 0$. All flow has to be routed through E_1. It follows that

$$f^Q(E_1^+) - f^Q(E_1^-) = d_S^Q \quad \text{and} \quad x(E_1) \geq \lceil \tfrac{d_S^Q}{c} \rceil.$$

Hence

$$cx(E_1) - (f^Q(E_1^+) - f^Q(E_1^-)) \geq c\lceil \tfrac{d_S^Q}{c} \rceil - d_S = d_S + c - r^Q - d_S = c - r^Q. \quad \text{(see Lemma 3.11)}$$

On the other hand if $x(\bar{E}_1) \geq 1$ then from $cx(E_1) - f^Q(E_1^+) + f^Q(E_1^-) \geq 0$ we conclude that

$$cx(E_1) + (c - r^Q)x(\bar{E}_1) + f^Q(E_1^-) - f^Q(E_1^+) \geq c - r^Q.$$

■

Note that if $d_S^Q < 0$ we get

$$cx(E_1) + \big(c - r\big)x(\bar{E}_1) + f^Q(E_1^+) - f^Q(E_1^-) \geq c - r, \tag{4.22}$$

with $r = r(|d_S^Q|, c)$, as a valid inequality for CS^{BI} and CS^{UN}. This again follows after multiplying all flow conservation constraints by -1.

To be able to easily generalise the inequality (4.21) to the multi-facility case, it is of interest to express it as a *MIR* inequality. A base inequality and some positive integer λ has to be found, such that (4.21) reduces to the corresponding $\frac{1}{\lambda}$-*MIR* inequality.

Unfortunately, it turns out to be difficult to find such a base inequality and λ. The author conjectures that they simply do not exist.

A necessary condition for cut inequalities Finally, consider the two cut inequalities, that we get from (4.18) and (4.19):

$$x(E_S) \geq \lceil \tfrac{d_S^{K+}}{c} \rceil,$$
$$x(E_S) \geq \lceil \tfrac{|d_S^{K-}|}{c} \rceil.$$

Thus

$$x(E_S) \geq \max(\lceil \tfrac{d_S^{K+}}{c} \rceil, \lceil \tfrac{|d_S^{K-}|}{c} \rceil) \tag{4.23}$$

is valid for CS^{BI} and CS^{UN}.

We will see that (4.23) can be facet-defining. In Lemma 4.8 it was shown that the cut inequality for CS^{DI} is not facet-defining when $d_S^Q < c$, $|A_S^+| > 1$ and $A_S^- = \emptyset$. There is a similar result for the undirected counterpart (4.23). But the key to prove it are the new cut set inequalities (4.21):

Lemma 4.20 *If the cut inequality* (4.23) *is facet-defining for CS^{BI} (or CS^{UN}) and $|E_S| > 1$, then $\max(d_S^{K+}, |d_S^{K-}|) > c$.*

Proof. Assume $d_S^{K+} \geq |d_S^{K-}|$ w. l. o. g.. The case $d_S^{K+} = c$ was discussed in Lemma 4.17. Suppose $d_S^{K+} < c$. Hence $d_S^{K+} = r^{K+}$ and $\eta^{K+} = 1$. Choose $E_1 \subset E_S$ such that $E_1, \bar{E}_1 \neq \emptyset$. Then with Theorem 4.19

$$cx(E_1) + (c - r^{K+})x(\bar{E}_1) + f^Q(E_1^-) - f^Q(E_1^+) \geq c - r^{K+}$$

and

$$cx(\bar{E}_1) + (c - r^{K+})x(E_1) + f^Q(\bar{E}_1^-) - f^Q(\bar{E}_1^+) \geq c - r^{K+}$$

are both valid inequalities for CS^{BI} (and CS^{UN}) different from the flow conservation constraint (4.1). Adding them up gives

$$(2c - r^{K+})x(E_S) + f^Q(E_S^-) - f^Q(E_S^+) \geq 2c - 2r^{K+}$$
$$\iff$$
$$(2c - r^{K+})x(E_S) - d_S^{K+} \geq 2c - 2r^{K+} \iff x(E_S) \geq 1 = \eta^{K+}.$$

Hence (4.23) is a sum of non-trivial valid inequalities when $|E_S| > 1$ and $d_S^{K+} < c$. ∎

Summary We have developed strong valid cut set inequalities for the cut sets CS^{BI} and CS^{UN}. By applying the same *MIR* procedure we have been able to state a class of flow cut inequality (4.18) analogous to (4.14) for CS^{DI}, introduced by Chopra et al. [1998]. We have specified necessary conditions for them to be facet-defining and have shown that in contrast to Theorem 4.10 they do not suffice to completely describe the cut sets in the single-commodity, single-facility case.

A complete description may be obtained by additionally considering a new class of valid cut set inequalities (4.21).

4.2.2.2 Cut set inequalities and sufficient conditions

In this section sufficient conditions for cut set inequalities of type (4.18) and of type (4.21) to be facet-defining will be provided.

We will mainly concentrate on the cut set CS^{BI} but all of the results in this section also hold for CS^{UN} with $K^- = \emptyset$ and some additional assumptions. The results will be formulated for CS^{BI}, we will prove them and will then discuss which modifications have to be carried out to make the statements true for CS^{UN}.

The cut inequality Cut inequalities for CS^{BI} were studied by Bienstock & Günlük [1996]. Considering two facilities, they state a result quite similar to the following one. Some parts of the proof are from their article. Moreover, it is similar to the the proof of Theorem 4.9. Even so, the proof is stated here to show the difference between facet-proofs for cut sets based on undirected supply graphs and those based on directed supply graphs.

Theorem 4.21 *The cut inequality* (4.23) *is facet-defining for CS^{BI} if and only if $r^{K+} < c$ and one of the following conditions holds*

i) $\max(d_S^{K+}, |d_S^{K-}|) > c$

ii) $|E_S| = 1$

Proof. Necessity: see Lemma 4.17 and Lemma 4.20.
Sufficiency: We assume $d_S^{K+} \geq |d_S^{K-}|$ w. l. o. g.. Set $\eta^{K+} := \lceil \frac{d_S^{K+}}{c} \rceil$. Remember that

$$c\eta^{K+} = d_S^{K+} + c - r^{K+} \quad \text{(Lemma 3.11).}$$

It has to be proven that $x(E_S) \geq \eta^{K+}$ defines a facet of CS^{BI}. To do so, we will show that the related face

$$F_{BI} = \{ (f, x) \in CS^{BI} : x(E_S) = \eta^{K+} \}$$

is nontrivial. Then by contradiction, we will show that it defines a facet. Consider $e_0 = uv \in E_S$. To construct a point $p_0 = (\bar{f}, \bar{x}) \in \mathbb{R}_+^{|E_S|} \times \mathbb{Z}_+^{2|E_S||K|}$ on the face F_{BI} define:

$$\bar{x}_{e_0} = \eta^{K^+}, \qquad\qquad \bar{f}^k_{uv} = d^k_S,\ \bar{f}^k_{vu} = 0 \quad \forall k \in K^+,$$
$$\bar{f}^k_{uv} = 0, \quad \bar{f}^k_{vu} = 0\ \forall k \in K^0, \qquad \bar{f}^k_{uv} = 0, \quad \bar{f}^k_{vu} = d^k_S\ \forall k \in K^-$$

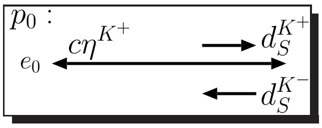


Figure 4.7: All flow is routed on e_0.

and fix the rest of the variables to zero. This means that we send all flow for K^+ on uv and all flow for K^- on vu after installing sufficient capacity on e_0. The point p_0 fulfils the flow conservation constraints for every $k \in K$ since every demand is satisfied. It meets the capacity constraints for CS^{BI} because

$$\sum_{k \in K} \bar{f}^k_{uv} = \sum_{k \in K^+} d^k_S = d^{K^+}_S < d^{K^+}_S + c - r^{K^+} = c\eta^{K^+} = c\bar{x}_{e_0}$$

and $|d^{K^-}_S| \leq d^{K^+}_S < \eta^{K^+}$. Since $\bar{x}(E_S) = \eta^{K^+}$, the point p_0 is on the face F_{BI}. Modifying p_0 by setting $x_{e_0} = \eta^{K^+}+1$ gives a point that is in CS^{BI} but not on the face F_{BI}. Hence $\emptyset \neq F_{BI} \neq CS^{BI}$.

It remains to show that F_{BI} is inclusion-wise maximal. This will be done by contradiction. Suppose there is a face F of CS^{BI} with $F_{BI} \subset F$. Let F be defined by

$$\beta^T x + \gamma^T f = \pi \tag{4.24}$$

where β, γ are vectors of appropriate dimension and $\pi \in \mathbb{R}$. We will show that (4.24) is a multiple of $x(E_S) = \eta^{K^+}$ up to a linear combination of flow conservation constraints, which contradicts $F_{BI} \subset F$.

Since multiples of the $|K|$ flow conservation constraints may be added to (4.24) without changing the induced face, $\gamma^k_{uv} = 0$ for all $k \in K$ with respect to the edge $e_0 = uv$ can be assumed.

Now, since capacity on e_0 is not saturated ($d^{K^+}_S < c\eta^{K^+}$), we can modify p_0 by simultaneously increasing flow for $k \in K$ on uv and vu by a small amount. This can be done for every commodity $k \in K$ without violating neither a flow conservation constraint nor a capacity constraint, so the points modified this way are still on the face F_{BI}. We can conclude that $\gamma^k_{vu} = 0$ for all $k \in K$.

Now we have to use the conditions of the theorem. If $|E_S| = 1$ we are finished. If otherwise $|E_S| \geq 2$ and $d^{K^+}_S > c$, we choose $e = ij \in E_S$ different from e_0. Next we construct a new point p from p_0 by setting $\bar{x}_{e_0} = \eta^{K^+} - 1$ and $\bar{x}_e = 1$.

The maximum amount of flow that can be routed on e_0 now is $c(\eta^{K^+} - 1) = d^{K^+}_S - r^{K^+}$. Note that from $d^{K^+}_S > c$ follows $d^{K^+}_S > r^{K^+}$ and $\eta^{K^+} - 1 \geq 1$.

We reroute a total flow of r^{K^+} such that

$$\bar{f}^k_{uv}, \bar{f}^k_{ij} > 0 \quad \text{and} \quad f^k_{vu}, \bar{f}^k_{ji} = 0\ \forall k \in K^+,$$
$$\bar{f}^k_{uv}, \bar{f}^k_{ij} = 0 \quad \text{and} \quad \bar{f}^k_{vu}, \bar{f}^k_{ji} > 0\ \forall k \in K^-$$

and the capacity on e is not saturated. This is possible because $d^{K^+}_S > r^{K^+}$ and $r^{K^+} < c$. (For a detailed description of such a rerouting see Definition and Lemma B.1). The point p is on the face because we did not change the total capacity on E_S and all demands as well as capacity constraints are still satisfied.

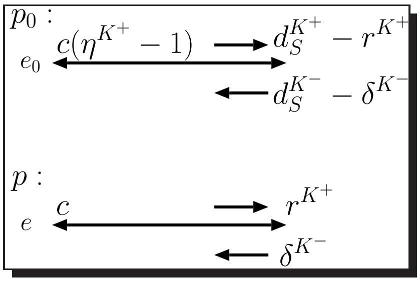


Figure 4.8: e is used to reroute the flow. ($\delta^{K^-} \leq r^{K^+}$)

First we modify this new point p by decreasing flow for $k \in K^+$ on uv and increasing flow on ij. The same can be done on vu and ji for $k \in K^-$. When having decreased flow on uv for $k \in K^+$, we can increase it for $k \in K^0$ on uv and ji. It follows that $\gamma_{ij}^k = 0$ for all $k \in K^+$ and $\gamma_{ji}^k = 0$ for all $k \in K^- \cup K^0$. Next we modify p by simultaneously increasing flow on ij and ji, which results in $\gamma_{ji}^k = 0$ for all $k \in K^+$ and $\gamma_{ij}^k = 0$ for all $k \in K^- \cup K^0$. Now plugging in the two constructed points p_0 and p into (4.24) we get

$$\beta_{e_0}\eta^{K^+} = \pi \quad \text{and} \quad \beta_{e_0}(\eta^{K^+} - 1) + \beta_e = \pi.$$

Hence $\beta_{e_0} = \beta_e$. Since e is arbitrary, we can conclude that $\gamma = 0$ and $\beta_e = \beta_{e_0}$ for all $e \in E_S$.

We have shown that (4.24) is a multiple of $x(E_S) \geq \eta^{K^+}$ plus a linear combination of flow conservation constraints. It follows that F and F_{BI} induce the same face, which is a contradiction. Hence F_{BI} is inclusion-wise maximal and together with $\emptyset \neq F_{BI} \neq CS^{BI}$ it defines a facet of CS^{BI}. ∎

Corollary 4.22 *The cut inequality*

$$x(E_S) \geq \lceil \frac{d_S^{K^+}}{c} \rceil = \eta^{K^+}$$

is facet-defining for CS^{UN} if and only if $r^{K^+} < c$ and one of the following conditions holds

i) $d_S^{K^+} > c$

ii) $|E_S| = 1$.

Proof. Theorem 4.21 holds for CS^{UN} with $K^- = \emptyset$. If a feasible point for CS^{BI} additionally meets the UNdirected capacity constraints, it is feasible for CS^{UN}.

The point p_0 in the proof fulfils the UNdirected capacity constraints because there is no flow on vu. The same is true for the point p, there is no flow on ji. In the remainder of the proof we perturb flow by small epsilons which does not affect the validity of (4.12). ∎

General flow cut inequalities We have shown under which conditions cut inequalities are facet-defining for CS^{BI} and will now state sufficient conditions for general flow cut inequalities of type (4.18) to be facet-defining for CS^{BI}. The key to the proof of Theorem 4.21 was the construction of the point p by rerouting, such that the total flow on edge e_0 does not exceed a value of $c(\eta^{K^+} - 1)$. Similar constructions will be used in the proofs of all the following facet theorems. A formalisation of such constructions can be found in Definition and Lemma B.1. Since the skeleton of those proofs is identical to the proof of Theorem 4.21 and because they are quite technical in the details, it was decided to put them to Appendix B.

Bienstock & Günlük [1996] investigate simple flow cut inequalities ($E_2 = \emptyset$) of the form (4.18) for the polyhedron NDP^{BI} with two facilities. They show under which conditions (4.18) with $E_2 = \emptyset$ is facet-defining. The following two theorems together with Theorem 4.4 can be seen as a generalisation of their work and results. Bienstock & Günlük [1996] state two more classes of flow cut inequalities in their article, both handling the case the demand being fractional, which is not considered here.

Theorem 4.23 *Let* $E_1, E_2 \subseteq E_S$ *and* $Q^+ \subseteq K^+$. *Assume* $r^{Q^+} < c$. *Setting* $Q = Q^+$, *the flow cut inequality* (4.18)

$$f^{Q^+}(\bar{E}_1^+) + cx(E_2) - f^{Q^+}(E_2^-) + r^{Q^+}\big(x(E_1) - x(E_2)\big) \geq r^{Q^+}\eta^{Q^+}$$

is facet-defining for CS^{BI} *if* $E_1 \backslash E_2 \neq \emptyset$ *and* $\bar{E}_1 \backslash E_2 \neq \emptyset$ *and one of the following conditions holds:*

i) $E_2 = \emptyset$ *and* $|Q^+| = 1$

ii) $E_2 = \emptyset$ *and* $d_S^{Q^+} > c$

iii) $E_2 \neq \emptyset$

Proof. see Appendix B.1 ∎

Corollary 4.24 *The flow cut inequality*

$$f^{Q^+}(\bar{E}_1^+) + cx(E_2) - f^{Q^+}(E_2^-) + r^{Q^+}\big(x(E_1) - x(E_2)\big) \geq r^{Q^+}\eta^{Q^+}$$

with $Q^+ \subseteq K^+$ *is facet-defining for* CS^{UN} *if* $E_1 \backslash E_2 \neq \emptyset$ *and* $\bar{E}_1 \backslash E_2 \neq \emptyset$ *and one of the following conditions holds:*

i) $E_2 = \emptyset$ *and* $|Q^+| = 1$

ii) $E_2 = \emptyset$ *and* $d_S^{Q^+} > c$

iii) $E_2 \neq \emptyset$ *and* $\boldsymbol{E_1 \cap E_2 = \emptyset}$

Proof. If $K^- = \emptyset$, then from the points constructed in the proof of Theorem 4.23 only those defined with edges in $E_1 \cap E_2$ do not meet the UNdirected capacity constraints (4.12). Recall that $E_1 \cap E_2 = \emptyset$ is a necessary condition for flow cut inequalities of type (4.18) to be facet-defining for CS^{UN}, see Lemma 4.17. ∎

The following theorem is an extension to Theorem 4.23. When $Q = K^+$ and $d_S^{K^+} \geq |d_S^{K^-}|$ there are more facet-defining flow cut inequalities of type (4.18). We can additionally handle the case $\bar{E}_1 \backslash E_2 = \emptyset$.

Theorem 4.25 *Let* $E_1, E_2 \subseteq E_S$. *Assume* $E_1 \backslash E_2 \neq \emptyset$ *and* $\boldsymbol{\bar{E}_1 \backslash E_2 = \emptyset}$, $r^{K^+} < c$ *and* $\boldsymbol{d_S^{K^+} \geq |d_S^{K^-}|}$. *Setting* $Q = K^+$ *the flow cut inequality* (4.18)

$$f^{K^+}(\bar{E}_1^+) + cx(E_2) - f^{K^+}(E_2^-) + r^{K^+}(x(E_1) - x(E_2)) \geq r^{K^+}\eta^{K^+}$$

is facet-defining for CS^{BI} *if one of the following conditions holds:*

i) $E_1 \cap E_2 = \emptyset$ *and* $E_2 \neq \emptyset$

ii) $E_1 \cap E_2 \neq \emptyset$ *and* $K^- \cup K^0 = \emptyset$

iii) $E_1 \cap E_2 \neq \emptyset$ *and* $K^0 = \emptyset$ *and* $d_S^{K^+} > |d_S^{K^-}|$ *and* $d_S^{K^+} > c$

Proof. see Appendix B.2. The main difference to the proof of the Theorem 4.23 is the construction of the starting point u_{e_0}. There is no edge $\bar{e}_0 = \bar{u}\bar{v}$ in $\bar{E}_1 \backslash E_2$ to route the flow for K^-. This flow is routed on vu instead, which is possible since $d_S^{K^+} \geq |d_S^{K^-}|$. ∎

It might be possible to construct a proof for the last theorem without the restrictions $K^0 = \emptyset$ and $d_S^{K+} > c$ in ii) and iii).

Corollary 4.26 *Let $E_1, E_2 \subseteq E_S$. Assume $r^{K+} < c$. The flow cut inequality*

$$f^{K+}(\bar{E}_1^+) + cx(E_2) - f^{K+}(E_2^-) + r^{K+}(x(E_1) - x(E_2)) \geq r^{K+}\eta^{K+}$$

is facet-defining for CS^{UN} if $E_1, E_2 \neq \emptyset$ and $E_2 = \bar{E}_1$.

Proof. This is Theorem 4.25 with the additional assumptions $K^- = \emptyset$ and $\boldsymbol{E_1 \cap E_2 = \emptyset}$. The point u_{e_0} is feasible for CS^{UN} with $K^- = \emptyset$ (see Definition and Lemma B.1). To ensure that the rest of the constructed points fulfil the UNdirected capacity constraints (4.12) we additionally need that $E_1 \cap E_2 = \emptyset$. ∎

At least in the single-commodity, single-facility case, we have now all sufficient and necessary conditions for flow cut inequalities of type (4.18) to be facet-defining.

Theorem 4.27 *In the single-commodity case, that is $|K^+| = 1$, $K^- \cup K^0 = \emptyset$ and $d_S = d_S^{K+}$, the flow cut inequality* (4.18)

$$f(\bar{E}_1^+) + cx(E_2) - f(E_2^-) + r(d_S, c)\big(x(E_1) - x(E_2)\big) \geq r(d_S, c)\lceil \tfrac{d_S}{c} \rceil$$

is facet-defining for CS^{BI} if and only if $r(d_S, c) < c$, $E_1 \backslash E_2 \neq \emptyset$ and one of the following conditions holds:

i) $E_2 = \emptyset$ *and* $\bar{E}_1 \neq \emptyset$

ii) $E_2 = \emptyset$ *and* $\bar{E}_1 = \emptyset$ *and* $d_S > c$

iii) $E_2 = \emptyset$ *and* $\bar{E}_1 = \emptyset$ *and* $|E_S| = 1$

iv) $E_2 \neq \emptyset$

Proof. This follows from Lemma 4.17, Lemma 4.20, Theorem 4.21, Theorem 4.23 and Theorem 4.25. ∎

Corollary 4.28 *In the single-commodity case, the flow cut inequality*

$$f(\bar{E}_1^+) + cx(E_2) - f(E_2^-) + r(d_S, c)\big(x(E_1) - x(E_2)\big) \geq r(d_S, c)\lceil \tfrac{d_S}{c} \rceil$$

is facet-defining for CS^{UN} if and only if $r(d_S, c) < c$, $E_1 \neq \emptyset$ and one of the following conditions holds:

i) $E_2 = \emptyset$ *and* $\bar{E}_1 \neq \emptyset$

ii) $E_2 = \emptyset$ *and* $\bar{E}_1 = \emptyset$ *and* $d_S > c$

iii) $E_2 = \emptyset$ *and* $\bar{E}_1 = \emptyset$ *and* $|E_S| = 1$

iv) $E_2 \neq \emptyset$ *and* $\boldsymbol{E_1 \cap E_2 \neq \emptyset}$

Proof. This follows from Lemma 4.17, Lemma 4.20 Corollary 4.22, Corollary 4.24 and Corollary 4.26. ∎

Example 4.16 (continued) *The inequalities* (4.20a), (4.20c), (4.20e), (4.20f) *and* (4.20i) *are facet-defining for CS^{BI} and CS^{UN}. Inequalities* (4.20j) *and* (4.20k) *define facets of CS^{BI} only, because* $E_1 \cap E_2 \neq \emptyset$.

A new class of cut set inequalities Until now we have only considered flow cut inequalities. In the following we will state a facet proof for the cut set inequalities of type (4.21). Examples show that they are often weak when Q is a proper subset of K^+ (or K^-). But for $Q = K^+$ and $d_S^{K^+} \geq |d_S^{K^-}|$ it can be proven that they are facet-defining.

Remember that the cut set inequalities of type (4.21) have no analogue for CS^{DI}. They reflect the special structure of BIdirected and UNdirected cut sets (at least in the single-facility case).

Theorem 4.29 *Let* $d_S^{K^+} \geq |d_S^{K^-}|$. *The cut set inequality* (4.21) *with* $Q = K^+$

$$cx(E_1) + \big(c - r^{K^+}\big)x(\bar{E}_1) + f^{K^+}(E_1^-) - f^{K^+}(E_1^+) \geq c - r^{K^+}$$

defines a nontrivial facet of CS^{BI} *if and only if* $r^{K^+} < c$ *and one of the following conditions holds:*

i) $E_1, \bar{E}_1 \neq \emptyset$

ii) $E_1 = \emptyset$ *and* $d_S^{K^+} < c$ *and* $|E_S| = 1$

iii) $\bar{E}_1 = \emptyset$ *and (*$d_S^{K^+} > c$ *or* $|E_S| = 1$*)*

Proof. Necessity: If $r^{K^+} = c$ then inequality (4.21) reduces to

$$cx(E_1) - (f^{K^+}(E_1^+) - f^{K^+}(E_1^-)) \geq 0,$$

which is the sum of capacity constraints and non-negativity constraints, thus it is trivial or not a facet. Assume $r^{K^+} < c$ in the sequel.

Suppose $E_1 = \emptyset$. Inequality (4.21) reduces to $x(E_S) \geq 1$, which is dominated by the cut inequality $x(E_S) \geq \eta^{K^+}$ if $d_S^{K^+} > c$.

If on the other hand $d_S^{K^+} < c$ and $|E_S| > 1$, then $x(E_S) \geq 1$ is the sum of two valid inequalities (see Lemma 4.20).

Now suppose that $\bar{E}_1 = \emptyset$. We can write (4.21) as

$$cx(E_S) + f^{K^+}(E_S^-) - f^{K^+}(E_S^+) \geq c - r^{K^+} \iff cx(E_S) \geq d_S^{K^+} + c - r^{K^+} = c\eta^{K^+}$$

the cut inequality (4.23) again, which is the sum of valid inequalities when $d_S^{K^+} < c$ and $|E_S| > 1$ (see Lemma 4.20).

Sufficiency: If ($E_1 = \emptyset$ and $d_S^{K^+} < c$), then (4.21) reduces to the cut inequality (4.23) which is facet-defining for CS^{BI} if $|E_S| = 1$.

The same happens when $\bar{E}_1 = \emptyset$, (4.21) reduces to the cut inequality (4.23), which is facet-defining if $|E_S| = 1$ or $d_S^{K^+} > c$ (see Theorem 4.21).

For the rest of the proof, which can be found in Appendix B.3, we assume that $E_1, \bar{E}_1 \neq \emptyset$. ∎

Corollary 4.30 *The cut set inequality* (4.21)

$$cx(E_1) + \big(c - r^{K^+}\big)x(\bar{E}_1) + f^{K^+}(E_1^-) - f^{K^+}(E_1^+) \geq c - r^{K^+}$$

defines a nontrivial facet of CS^{UN} *if and only if* $r^{K^+} < c$ *and one of the following conditions holds:*

i) $E_1, \bar{E}_1 \neq \emptyset$

ii) $E_1 = \emptyset$ *and* $d_S^{K+} < c$ *and* $|E_S| = 1$

iii) $\bar{E}_1 = \emptyset$ *and (*$d_S^{K+} > c$ *or* $|E_S| = 1$*)*

Proof. Again with $K^- = \emptyset$ all constructed points in the proof of Theorem 4.29 fulfil the UNdirected capacity constraints. ∎

Example 4.16 (continued)

$$3x_1 + 2x_2 + f_3 - f_1 \geq 2$$

and

$$3x_2 + 2x_1 + f_4 - f_2 \geq 2$$

are facet-defining inequalities for CS^{BI} *and* CS^{UN} *of the form* (4.21)*. Together with all flow cut inequalities they describe all nontrivial facets of both polyhedra in this example (and many others). It can be presumed, that cut set inequalities of type* (4.18) *and* (4.21) *(together with all trivial facets) suffice to give a complete description of* CS^{BI} *and* CS^{UN} *in the single-commodity case, analogous to Theorem 4.10 (Atamtürk [2002]).*

For completeness all nontrivial facet-defining inequalities of CS^{BI}*,* CS^{UN} *are stated here again: Flow cut inequalities of type* (4.18)*:*

$$
\begin{aligned}
f_2 + x_1 &\geq 3 && (E_1 = \{1\},\ E_2 = \emptyset) && \textit{simple flow cut inequality}\\
f_2 + 2x_2 - f_4 + x_1 &\geq 3 && (E_1 = \{1\},\ E_2 = \{2\}) &&\\
f_1 + x_2 &\geq 3 && (E_1 = \{2\},\ E_2 = \emptyset) && \textit{simple flow cut inequality}\\
f_1 + 2x_1 - f_3 + x_2 &\geq 3 && (E_1 = \{2\},\ E_2 = \{1\}) &&\\
x_1 + x_2 &\geq 3 && (E_1 = \{1,2\},\ E_2 = \emptyset) && \textit{cut inequality}\\
3x_1 - f_3 + x_2 &\geq 3 && (E_1 = \{1,2\},\ E_2 = \{1\}) && \textit{facet only of } CS^{BI}\\
3x_2 - f_4 + x_1 &\geq 3 && (E_1 = \{1,2\},\ E_2 = \{2\}) && \textit{facet only of } CS^{BI}
\end{aligned}
$$

Flow cut inequalities of type (4.21)*:*

$$
\begin{aligned}
3x_1 + 2x_2 + f_3 - f_1 &\geq 2 && (E_1 = \{1\},\ \bar{E}_1 = \{2\})\\
3x_2 + 2x_1 + f_4 - f_2 &\geq 2 && (E_1 = \{2\},\ \bar{E}_1 = \{1\})
\end{aligned}
$$

Summary In Section 4.2 the facial structure of the cut sets CS^{DI}, CS^{BI} and CS^{UN} has been investigated. We started with CS^{DI}, summarised and even supplemented results of Atamtürk [2002]. A large class of cut set inequalities called flow cut inequalities has been introduced and facet theorems have been stated.

By simply using the same *MIR* procedure in Section 4.2.2 it has been shown that there is an obvious analogue for flow cut inequalities in the BIdirected and UNdirected case. Flow cut inequalities generalise already known cut inequalities and simple flow cut inequalities.

In contrast to CS^{DI} they do not suffice to give a complete description in the single-commodity, single-facility case, which is reflected by a second class of cut set inequalities for CS^{BI} and CS^{UN}. For both classes of cut set inequalities we have presented facet proofs.

In the following section we will concentrate on the general multi-facility case.

4.3 The cut set for multi-facility problems

In this section the results for cut sets with a single design variable will be extended to cut sets with more than one facility to install. We will fix all but one design variables to their lower bound zero. For the resulting single-facility restrictions we know facet-defining inequalities from Section 4.2, which will be lifted by using the subadditive *MIR* functions defined in Chapter 3.

The polyhedra CS^{DI}, CS^{BI} and CS^{UN} for the general case $|T| \in \mathbb{Z}_+ \backslash \{0\}$ were already introduced in Section 4.1. Assume that $|T| \geq 2$. Atamtürk [2002] states the exact lifting function for flow cut inequalities of cut sets with DIrected capacity constraint. In Section 4.3.1 it will be shown that it equals a certain *MIR* function which proves that lifting with *MIR* is exact. This observation motivates a general *MIR*-procedure that produces strong valid flow cut inequalities in the multi-facility case for all three capacity models DIrected, BIdirected and UNdirected.

4.3.1 DIrected capacity constraints

Aggregating and substituting as in Section 4.2.1 gives the following valid base inequality for the set CS^{DI} similar to (4.13)

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + \sum_{t \in T} c^t \big(x^t(A_1^+) - x^t(A_2^-)\big) \geq d_S^Q. \tag{4.25}$$

where $\bar{f}^Q(A_2^-) = \sum_{t \in T} c^t x^t(A_2^-) - f^Q(A_2^-) \geq 0$ and A_1^+, A_2^- are subsets of $A_S^+ = \delta^+(S), A_S^- = \delta^-(S)$, respectively.

Given the facility $s \in T$, let CS_s^{DI} be the restriction of CS^{DI} obtained by fixing all design variables x_a^t, with $a \in A_S$ and $t \in T \backslash \{s\}$, to their lower bound zero. Hence

$$CS_s^{DI} = \text{conv}\{\, (f, x) \in CS^{DI} : \; x_a^t = 0, a \in A_S, t \in T \backslash \{s\} \,\}.$$

From Proposition 4.6 follows that

$$f^Q(\bar{A}_1^+) + c^s x^s(A_2^-) - f^Q(A_2^-) + r_s^Q \big(x^s(A_1^+) - x^s(A_2^-)\big) \geq r_s^Q \eta_s^Q \tag{4.26}$$

is a valid single-facility flow cut inequality for CS_s^{DI}, where $r_s^Q = r(d_S^Q, c^s)$ and $\eta_s^Q = \lceil \frac{d_S^Q}{c^s} \rceil$. From Lemma 4.8, Theorem 4.9, 4.10, 4.11 and Corollary 4.13 we know necessary and sufficient conditions for (4.26) being facet-defining for CS_s^{DI}.

Lifting The flow cut inequality (4.26) is a $\frac{1}{c^s}$-*MIR* inequality as shown in Section 4.2.1. Hence with Proposition 3.14 the subadditive function $\mathcal{G}_{d_S^Q, c^s}$ defines an upper bound on the exact lifting function and can be used for lifting (4.26) to a valid inequality of CS^{DI} (see Section 3.2, Proposition 3.14).

Atamtürk [2002] calculates the exact lifting function and shows that under certain additional conditions exact lifting of (4.26) to a valid inequality of CS^{DI} can be done simultaneously, because the exact lifting function is subadditive. He states the lifted inequality as

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + \sum_{t \in T} \phi_s^+(c^t) x^t(A_1^+) + \sum_{t \in T} \phi_s^-(c^t) x^t(A_2^-) \geq r_s^Q \eta_s^Q, \tag{4.27}$$

where

$$\phi_s^+(c^t) = \begin{cases} c^t - k(c^s - r_s^Q) & \text{if } kc^s \le c^t < kc^s + r_s^Q \\ (k+1)r_s^Q & \text{if } kc^s + r_s^Q \le c^t < (k+1)c^s \end{cases} \qquad k \text{ integer}$$

and

$$\phi_s^-(c^t) = \begin{cases} c^t - (k-1)r_s^Q & \text{if } (k-1)c^s \le c^t < kc^s - r_s^Q \\ k(c^s - r_s^Q) & \text{if } kc^s - r_s^Q \le kc^t < c^s \end{cases} \qquad k \text{ integer}$$

Note that the integer k is unique since $0 \le r_s^Q \le c^s$.

Set $d := d_S^Q$. In the following we will show that inequality (4.27) can be seen as the $\frac{1}{c^s}$-*MIR* inequality for the base inequality (4.25). This gives an alternative proof for the validity of (4.27) and implies that the subadditive lifting function used for lifting (4.26) to (4.27) is the *MIR* function $\mathcal{G}_{d,c^s}$.

It turns out that

$$\phi_s^+(c^t) = \mathcal{G}_{d,c^s}(c^t),$$

which provides an alternative description for $\mathcal{G}_{d,c^s}$.

Theorem 4.31 *Inequality* (4.27) *is the* $\frac{1}{c^s}$-MIR *inequality for the base inequality* (4.25).

Proof. Rewriting (4.27) gives

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + \sum_{t\in T} \phi_s^+(c^t)x^t(A_1^+) + \sum_{t\in T}(\phi_s^-(c^t) - c^t)x^t(A_2^-) \ge r_s^Q\eta_s^Q.$$

The $\frac{1}{c^s}$-*MIR* inequality for (4.25) is

$$\overline{\mathcal{G}}_{d,c^s}(1)f^Q(\bar{A}_1^+) + \overline{\mathcal{G}}_{d,c^s}(1)\bar{f}^Q(A_2^-) + \sum_{t\in T}\mathcal{G}_{d,c^s}(c^t)x^t(A_1^+) + \sum_{t\in T}\mathcal{G}_{d,c^s}(-c^t)x^t(A_2^-) \ge \mathcal{G}_{d,c^s}(d)$$

$$\iff f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + \sum_{t\in T}\mathcal{G}_{d,c^s}(c^t)x^t(A_1^+) + \sum_{t\in T}\mathcal{G}_{d,c^s}(-c^t)x^t(A_2^-) \ge r(d,c^s)\lceil \tfrac{d}{c^s}\rceil = r_s^Q\eta_s^Q.$$

It remains to show that

$$\textbf{1. } \phi_s^+(c^t) = \mathcal{G}_{d,c^s}(c^t) \quad \text{and} \quad \textbf{2. } (\phi_s^-(c^t) - c^t) = \mathcal{G}_{d,c^s}(-c^t).$$

To prove this we make use of the equations

$$c^s\lceil \tfrac{c^t}{c^s}\rceil = c^t + c^s - r(c^t,c^s) \quad \text{and} \quad r(-c^t,c^s) = c^s - r(c^t,c^s)$$

several times. The latter is valid if $r(c^t,c^s) < c^s$, thus $\langle\frac{c^t}{c^s}\rangle > 0$. The validity of both is proven in Lemma 3.11 i) and 3.11 ii) respectively.

1. Set $k := \lceil\frac{c^t}{c^s}\rceil - 1$. First suppose that $r(d,c^s) > r(c^t,c^s)$. It follows that $r(c^t,c^s) < c^s$ and hence $k = \lfloor\frac{c^t}{c^s}\rfloor$. Using Lemma 3.11 i) yields

$$(\lceil\tfrac{c^t}{c^s}\rceil - 1)c^s = kc^s < c^t < c^t - r(c^t,c^s) + r(d,c^s) = c^s\lceil\tfrac{c^t}{c^s}\rceil - c^s + r(d,c^s) = kc^s + r(d,c^s).$$

Hence $\phi_s^+(c^t) = c^t - k(c^s - r(d,c^s)) = c^t - \lfloor \frac{c^t}{c^s} \rfloor (c^s - r(d,c^s))$. But again with Lemma 3.11 i)

$$\begin{aligned} c^t - \lfloor \tfrac{c^t}{c^s} \rfloor (c^s - r(d,c^s)) &= c^t - \lceil \tfrac{c^t}{c^s} \rceil (c^s - r(d,c^s)) + c^s - r(d,c^s) \\ &= r(d,c^s)\lceil \tfrac{c^t}{c^s} \rceil + c^t + c^s - c^s \lceil \tfrac{c^t}{c^s} \rceil - r(d,c^s) \\ &= r(d,c^s)\lceil \tfrac{c^t}{c^s} \rceil + r(c^t,c^s) - r(d,c^s) = \mathcal{G}_{d,c^s}(c^t). \end{aligned}$$

Now suppose that $r(d,c^s) \le r(c^t,c^s)$. Then

$$kc^s + r(d,c^s) \le (\lceil \tfrac{c^t}{c^s} \rceil - 1)c^s + r(c^t,c^s) = c^t \le \lceil \tfrac{c^t}{c^s} \rceil c^s = (k+1)c^s$$

and $\phi_s^+(c^t) = r(d,c^s)(k+1) = r(d,c^s)\lceil \frac{c^t}{c^s} \rceil = \mathcal{G}_{d,c^s}(c^t)$.

2. Set $k := \lfloor \frac{c^t}{c^s} \rfloor + 1$. First suppose that $r(d,c^s) > r(-c^t,c^s)$ implying that $c^s > r(-c^t,c^s) = c^s - r(c^t,c^s)$, $k = \lceil \frac{c^t}{c^s} \rceil$ and

$$kc^s - r(d,c^s) = \lceil \tfrac{c^t}{c^s} \rceil c^s - r(d,c^s) < \lceil \tfrac{c^t}{c^s} \rceil c^s + c^s - r(c^t,c^s) = c^t < \lceil \tfrac{c^t}{c^s} \rceil c^s = kc^s.$$

Thus

$$\begin{aligned} \phi_s^-(c^t) - c^t = k(c^s - r(d,c^s)) - c^t &= c^s \lceil \tfrac{c^t}{c^s} \rceil + r(d,c^s)\lfloor \tfrac{-c^t}{c^s} \rfloor - c^t \\ &= c^t + c^s - r(c^t,c^s) + r(d,c^s)\lceil \tfrac{-c^t}{c^s} \rceil - r(d,c^s) - c^t \\ &= r(d,c^s)\lceil \tfrac{-c^t}{c^s} \rceil - r(d,c^s) + r(-c^t,c^s) = \mathcal{G}_{d,c^s}(-c^t). \end{aligned}$$

If $r(d,c^s) \le r(-c^t,c^s) = c^s$ then

$$(k-1)c^s = c^t \le c^t + c^s - r(d,c^s) = (\lfloor \tfrac{c^t}{c^s} \rfloor + 1)c^s - r(d,c^s) = kc^s - r(d,c^s).$$

And if $r(d,c^s) \le r(-c^t,c^s) < c^s$ then

$$\begin{aligned} (k-1)c^s = \lfloor \tfrac{c^t}{c^s} \rfloor c^s \le c^t \le c^t + r(-c^t,c^s) - r(d,c^s) &= c^t + c^s - r(c^t,c^s) - r(d,c^s) \\ &= \lceil \tfrac{c^t}{c^s} \rceil c^s - r(d,c^s) = kc^s - r(d,c^s). \end{aligned}$$

It follows that

$$\phi_s^-(c^t) - c^t = -(k-1)r(d,c^s) = -r(d,c^s)\lfloor \tfrac{c^t}{c^s} \rfloor = r(d,c^s)\lceil \tfrac{-c^t}{c^s} \rceil = \mathcal{G}_{d,c^s}(-c^t).$$

The proof is complete. ∎

Atamtürk [2002] states conditions for (4.27) to be facet-defining for CS^{DI}. The following results are a consequence of Proposition 3.13 about superadditivity (or subadditivity) and lifting.

Proposition 4.32 (Atamtürk [2002]) *Suppose $r(d,c^s) < c$ and $|K| = |K^+| = |Q| = 1$. In the single-commodity case the multi-facility flow cut inequality* (4.27) *is facet-defining for CS^{DI} if $A_1^+, \bar{A}_1^+, A_2^- \neq \emptyset$.*

Proposition 4.33 (Atamtürk [2002]) *Suppose $r(d,c^s) < c$. The multi-facility flow cut inequality* (4.27) *is facet-defining for CS^{DI} if $A_1^+, \bar{A}_1^+, A_2^-, \bar{A}_2^- \neq \emptyset$.*

Strengthening Notice that both propositions exclude cut inequalities ($A_2^- = \emptyset$ and $\bar{A}_1^+ = \emptyset$) as well as simple flow cut inequalities ($A_2^- = \emptyset$). The reason is, that in those cases $\mathcal{G}_{d,c^s}$ is not the the exact lifting function (but still a valid subadditive upper bound). Atamtürk [2002] proposes a strengthening of the inequality (4.27) (hence a strengthening of the valid lifting function) if $A_2^- = \emptyset$. Set

$$\tilde{\phi}_s^+(c^t) = \begin{cases} \phi_s^+(c^t) = \mathcal{G}_{d,c^s}(c^t) & \text{if } c^t < d_S^Q \\ r_s^Q \eta_s^Q & \text{else .} \end{cases}$$

Note that if $c^t = d_S^Q$, then $\tilde{\phi}_s^+(c^t) = r_s^Q \eta_s^Q = \mathcal{G}_{d,c^s}(c^t)$. Another important observation is that $c^t > d_S^Q \iff \tilde{\phi}_s^+(c^t) < \phi_s^+(c^t)$ (see Atamtürk [2002]).

Proposition 4.34 (Atamtürk [2002]) *The strengthened simple flow cut inequality*

$$f^Q(\bar{A}_1^+) + \sum_{t \in T} \tilde{\phi}_s^+(c^t) x^t(A_1^+) \geq r_s^Q \eta_s^Q \tag{4.28}$$

is valid for CS^{DI} and at least as strong as (4.27) *if* $A_2^- = \emptyset$.

We prove this result by showing that (4.28) can be obtained by a second *MIR* step.

Proof.

$$f^Q(\bar{A}_1^+) + \sum_{t \in T} \phi_s^+(c^t) x^t(A_1^+) \geq r_s^Q \eta_s^Q \tag{4.29}$$

is a valid $\frac{1}{c^s}$-*MIR* inequality for CS^{DI} as already shown. Let $\bar{c} := \max(\phi_s^+(c^t))_{t \in T}$. If $\bar{c} \leq r_s^Q \eta_s^Q$, then $\tilde{\phi}_s^+(c^t) = \phi_s^+(c^t)$ for all $t \in T$. Suppose $\bar{c} > r_s^Q \eta_s^Q$. It follows that

$$r' := r(r_s^Q \eta_s^Q, \bar{c}) = r_s^Q \eta_s^Q \quad \text{and} \quad r(\phi_s^+(c^t), \bar{c}) = \phi_s^+(c^t).$$

This gives

$$\mathcal{G}_{r',\bar{c}}(\phi_s^+(c^t)) = r' - \left(r' - \phi_s^+(c^t)\right)^+ = \min(r', \phi_s^+(c^t)) = \tilde{\phi}_s^+(c^t).$$

Hence using (4.29) as the base inequality for the calculation of the $\frac{1}{\bar{c}}$-*MIR* inequality gives (4.28). ∎

The second *MIR* step is equivalent to rounding down all coefficients $\mathcal{G}_{d,c^s}(c^t)$ to the value of the right hand side $r_s^Q \eta_s^Q = r'$, if $\mathcal{G}_{d,c^s}(c^t) > r'$. Note that every coefficient of the left hand side is positive if $A_2^- = \emptyset$. The new inequality is at least as strong as the $\frac{1}{c^s}$-*MIR* inequality if $A_2^- = \emptyset$.

The *MIR* procedure to obtain strong valid flow cut inequalities for CS^{DI} in the multi-facility case can be summarised as follows:

Aggregating and Substituting Choose a commodity subset Q, subsets of the dicut arcs A_1^+ and A_2^- and a facility $s \in T$ such that the restricted flow cut inequality (4.26) defines a facet for CS_s^{DI}. Aggregate and substitute to arrive at (4.25). For necessary and sufficient conditions see Section 4.2.1.

Scaling and *MIR*

1. Calculate the $\frac{1}{c^s}$-*MIR* inequality (4.27) corresponding to (4.25) given by

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t)x^t(A_1^+) + \sum_{t\in T} \mathcal{G}_{d,c^s}(-c^t)x^t(A_2^-) \geq r_s^Q \eta_s^Q = r'.$$

2. If $A_2^- = \emptyset$ and $\bar{c} := \max(\mathcal{G}_{d,c^s}(c^t))_{t\in T} > r'$, then round down left hand side coefficients to r'. This gives

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + \sum_{t\in T} \min(r', \mathcal{G}_{d,c^s}(c^t))x^t(A_1^+) \geq r'$$

being at least as strong as the $\frac{1}{c^s}$-*MIR* from step 1.

Example 4.35 *Consider the following two-facility cut set with two outflow- and one inflow-arc. The two possible capacities to install are $c^1 = 2$ and $c^2 = 5$. A demand of $d = 3$ has to be satisfied.*

$$\begin{aligned} CS^{DI} = \text{conv}\{x \in \mathbb{Z}_+^6, f \in \mathbb{R}_+^3 \mid\ & f_1 + f_2 - f_3 = 3 \\ & f_i \leq 2x_i^1 + 5x_i^2, i \in \{1,2,3\}\} \end{aligned}$$

If $A_1^+ \neq \emptyset$ we can formulate the following 12 flow cut inequalities of type (4.27)*:*

$$f_2 + x_1^1 + 3x_1^2 \geq 2 \quad (A_1^+ = \{1\},\ A_2^- = \emptyset,\quad c^s = 2) \tag{4.30a}$$
$$f_2 + 2x_1^1 + 3x_1^2 \geq 3 \quad (A_1^+ = \{1\},\ A_2^- = \emptyset,\quad c^s = 5) \tag{4.30b}$$
$$f_2 - f_3 + x_1^1 + 3x_1^2 + x_3^1 + 3x_3^2 \geq 2 \quad (A_1^+ = \{1\},\ A_2^- = \{3\},\quad c^s = 2) \tag{4.30c}$$
$$f_2 - f_3 + 2x_1^1 + 3x_1^2 + 2x_3^1 + 2x_3^2 \geq 3 \quad (A_1^+ = \{1\},\ A_2^- = \{3\},\quad c^s = 5) \tag{4.30d}$$
$$f_1 + x_2^1 + 3x_2^2 \geq 2 \quad (A_1^+ = \{2\},\ A_2^- = \emptyset,\quad c^s = 2) \tag{4.30e}$$
$$f_1 + 2x_2^1 + 3x_2^2 \geq 3 \quad (A_1^+ = \{2\},\ A_2^- = \emptyset,\quad c^s = 5) \tag{4.30f}$$
$$f_1 - f_3 + x_2^1 + 3x_2^2 + x_3^1 + 3x_3^2 \geq 2 \quad (A_1^+ = \{2\},\ A_2^- = \{3\},\quad c^s = 2) \tag{4.30g}$$
$$f_1 - f_3 + 2x_2^1 + 3x_2^1 + 2x_3^1 + 2x_3^2 \geq 3 \quad (A_1^+ = \{2\},\ A_2^- = \{3\},\quad c^s = 5) \tag{4.30h}$$
$$x_1^1 + 3x_1^2 + x_2^1 + 3x_2^2 \geq 2 \quad (A_1^+ = \{1,2\},\ A_2^- = \emptyset,\quad c^s = 2) \tag{4.30i}$$
$$2x_1^1 + 3x_1^2 + 2x_2^1 + 3x_2^2 \geq 3 \quad (A_1^+ = \{1,2\},\ A_2^- = \emptyset,\quad c^s = 5) \tag{4.30j}$$
$$-f_3 + x_1^1 + 3x_1^2 + x_2^1 + 3x_2^2 + x_3^1 + 3x_3^2 \geq 2 \quad (A_1^+ = \{1,2\},\ A_2^- = \{3\}, c^s = 2) \tag{4.30k}$$
$$-f_3 + 2x_1^1 + 3x_1^2 + 2x_2^1 + 3x_2^2 + 2x_3^1 + 2x_3^2 \geq 3 \quad (A_1^+ = \{1,2\},\ A_2^- = \{3\}, c^s = 5) \tag{4.30l}$$

From Proposition 4.32 we know that the inequalities (4.30c), (4.30d), (4.30g) *and* (4.30h) *define facets for CS^{DI} because $A_1^+, \bar{A}_1^+, A_2^- \neq \emptyset$ holds for them. Using* PORTA *(Christof & Löbel [2005]) it can be seen that* (4.30b), (4.30f), (4.30k) *and* (4.30l) *are facet-defining either.*

The inequalities (4.30a), (4.30e) *and* (4.30i) *can be strengthened with Proposition 4.34 because $A_2^- = \emptyset$ and $\bar{c} = \max(\phi_s^+(c^t))_{t\in T} = 3 > 2 = r_s^Q \eta_s^Q$. Rounding down the coefficients of the left hand side gives the following two-step* MIR *inequalities*

$$f_2 + x_1^1 + 2x_1^2 \geq 2 \quad (A_1^+ = \{1\},\ A_2^- = \emptyset,\quad \textit{strengthened}) \tag{4.30a2}$$
$$f_1 + x_2^1 + 2x_2^2 \geq 2 \quad (A_1^+ = \{2\},\ A_2^- = \emptyset,\quad \textit{strengthened}) \tag{4.30e2}$$
$$x_1^1 + 2x_1^2 + x_2^1 + 2x_2^2 \geq 2 \quad (A_1^+ = \{1,2\},\ A_2^- = \emptyset, \textit{strengthened}) \tag{4.30i2}$$

all facet-defining for CS^{DI}.

Summary It was shown how to exploit the results of Section 4.2.1 to obtain strong valid inequalities for CS^{DI} in the general multi-commodity, multi-facility case.

Facet-defining flow cut inequalities for single-facility restrictions of CS^{DI} can be lifted to valid inequalities of CS^{DI}. We made use of the introduction to *MIR*, superadditivity and lifting given in Section 3.2 to show that under certain conditions the *MIR* function $\mathcal{G}_{d,c^s}$ equals the exact lifting function. A strengthening of the lifted inequalities was proposed if this is not the case. Motivated by these results we stated a *MIR* procedure that can be used to obtain strong valid inequalities for cut sets with DIrected capacity constraints.

4.3.2 BIdirected and UNdirected capacity constraints

We proceed similar to the last section. Facet-defining inequalities for single-facility restrictions of the cut sets CS^{BI} and CS^{UN} will be lifted using the same subadditive *MIR* function. In fact the only difference between the *MIR* procedure of CS^{DI}, CS^{BI} and CS^{UN} is, that it depends on the capacity model which inequalities are facet-defining for single-facility restrictions, as shown in Section 4.2.

With the arguments of Section 4.2.2 we assume $K^- = \emptyset$ for the cut set CS^{UN}. Aggregating and substituting as in Section 4.2.2 gives the following valid base inequality for the sets CS^{BI} and CS^{UN} similar to (4.17)

$$f^Q(\bar{E}_1^+) + \bar{f}^Q(E_2^-) + \sum_{t\in T} c^t\big(x^t(E_1) - x^t(E_2)\big) \geq d_S^Q, \tag{4.31}$$

where $\bar{f}^Q(E_2^-) = \sum_{t\in T} c^t x^t(E_2) - f^Q(E_2^-) \geq 0$ and E_1, E_2 are subsets of the cut E_S.

Given the facility $s \in T$, let CS_s^{BI} and CS_s^{UN} be the single-facility restrictions of CS^{BI} and CS^{UN} obtained by fixing all design variables x_e^t, with $e \in E_S$ and $t \in T\backslash\{s\}$, to their lower bound zero. Hence

$$\begin{aligned} CS_s^{BI} &= \operatorname{conv}\{\,(f,x) \in CS^{BI} :\ x_e^t = 0, e \in E_S, t \in T\backslash\{s\}\,\}.\\ CS_s^{UN} &= \operatorname{conv}\{\,(f,x) \in CS^{UN} :\ x_e^t = 0, e \in E_S, t \in T\backslash\{s\}\,\}.\end{aligned}$$

From Proposition 4.14 follows that

$$f^Q(\bar{E}_1) + c^s x^s(E_2) - f^Q(E_2^-) + r_s^Q\big(x^s(E_1) - x^s(E_2)\big) \geq r_s^Q \eta_s^Q \tag{4.32}$$

is valid for CS_s^{BI} and CS_s^{UN}, where $r_s^Q = r(d_S^Q, c^s)$ and $\eta_s^Q = \lceil \frac{d_S^Q}{c^s} \rceil$.

Lifting Set $d := d_S^Q$. For conditions for (4.32) being facet-defining for CS_s^{BI} or CS_s^{UN} see Section 4.2.2. Both inequalities are $\frac{1}{c^s}$-*MIR* inequalities. Hence with Proposition 3.14 the subadditive function $\mathcal{G}_{d,c^s}$ defines an upper bound on the exact lifting function and can be used for lifting (4.32) to valid inequalities of CS^{BI} and CS^{UN}, respectively (see Section 3.2).

The exact lifting functions are not known but motivated by the results of Atamtürk [2002] for DIrected cut sets (see Section 4.3.1), it can be conjectured that under certain additional assumptions the exact lifting function for (4.32) is given by $\mathcal{G}_{d,c^s}$. We use *MIR* here as a valid lifting procedure. Given the base inequality (4.31), the following flow cut inequality is valid for CS^{BI} and CS^{UN}:

$$f^Q(\bar{E}_1^+) + \bar{f}^Q(E_2^-) + \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t) x^t(E_1) + \sum_{t\in T} \mathcal{G}_{d,c^s}(-c^t) x^t(E_2) \geq r_s^Q \eta_s^Q. \tag{4.33}$$

Strengthening If $E_2 = \emptyset$, then all left hand side coefficients of (4.33) are positive and we can round down all coefficients of integer variables to the value of the right hand side if they are greater. Note that $r_s^Q \eta_s^Q \geq 1$. As shown in Section 4.3.1 this strengthening can be seen as a second *MIR* step. Applied to (4.33) we arrive at the simple flow cut inequality

$$f^Q(\bar{E}_1^+) + \sum_{t \in T} \min(r_s^Q \eta_s^Q, \mathcal{G}_{d,c^s}(c^t)) x^t(E_1) \geq r_s^Q \eta_s^Q. \tag{4.34}$$

It turns out that the *MIR* procedure, to derive strong valid flow cut inequalities for cut sets with BIdirected and UNdirected capacity constraints, is equivalent to the procedure in Section 4.3.1:

Aggregating and Substituting Choose a commodity subset Q, subsets of the cut arcs E_1, E_2 and a facility $s \in T$ such that the restricted flow cut inequality (4.32) defines a facets of CS_s^{BI} or CS_s^{UN}. For necessary and sufficient conditions see Section 4.2.2. Important necessary conditions were

$$E_1 \neq \emptyset \wedge E_1 \backslash E_2 \neq \emptyset$$

for CS_s^{BI} and

$$E_1 \neq \emptyset \wedge E_1 \cap E_2 = \emptyset$$

for CS_s^{UN} (Lemma 4.17). Aggregate and substitute to arrive at (4.31).

Scaling and *MIR*

1. Calculate the $\frac{1}{c^s}$-*MIR* inequality (4.33) corresponding to (4.31).
2. If $E_2 = \emptyset$ then round down left hand side coefficients to the value of the right hand side. This gives inequality (4.34).

Example 4.36 *Bienstock & Günlük [1996] consider network design polyhedra with BIdirected capacity constraints and two facilities, where $c^1 = 1$ and $c^2 = \lambda \in \mathbb{Z}_+, \lambda > 1$. Specialising* (4.33) *with $d = d_S^Q$, $c^s = \lambda$, $r = r(d, \lambda)$ and $\eta = \lceil \frac{d}{\lambda} \rceil$ gives*

$$f^Q(\bar{E}_1^+) + \bar{f}^Q(E_2^-) + \mathcal{G}_{d,\lambda}(1)x^1(E_1) + \mathcal{G}_{d,\lambda}(\lambda)x^2(E_1) + \mathcal{G}_{d,\lambda}(-1)x^1(E_2) + \mathcal{G}_{d,\lambda}(-\lambda)x^2(E_2) \geq r\eta$$

$$\Longleftrightarrow$$

$$f^Q(\bar{E}_1^+) - f^Q(E_2^-) + x^1(E_1) + rx^2(E_1) + (1 - r(\tfrac{d}{\lambda}))x^1(E_2) + (\lambda - r)x^2(E_2) \geq r\eta,$$

which reduces to the simple flow cut inequalities of Bienstock & Günlük [1996] if $E_2 = \emptyset$ and to the cut inequalities of Bienstock & Günlük [1996] if additionally $E_1 = E_S$. They state two more classes of simple flow cut inequalities, both corresponding to the case that d is fractional, which we do not consider here.

Magnanti & Mirchandani [1993] investigate network design polyhedra with UNdirected capacity constraints, three facilities and one commodity, where $c^1 = 1$, $c^2 = C \in \mathbb{Z}_+, C > 1$ and $c^3 = \lambda C \in \mathbb{Z}_+, \lambda > 1$. We can formulate two cut inequalities of type (4.33) *corresponding to $c^s = C$ and $c^s = \lambda C$, which are*

$$\mathcal{G}_{d,C}(1)x^1(E_S) + \mathcal{G}_{d,C}(C)x^2(E_S) + \mathcal{G}_{d,C}(\lambda C)x^3(E_S) \geq r_1 \lceil \tfrac{d}{C} \rceil$$

and

$$\mathcal{G}_{d,\lambda C}(1)x^1(E_S) + \mathcal{G}_{d,\lambda C}(C)x^2(E_S) + \mathcal{G}_{d,\lambda C}(\lambda C)x^3(E_S) \geq r_2\lceil \tfrac{d}{\lambda C}\rceil$$

reducing to the cut inequalities of Magnanti & Mirchandani [1993]:

$$x^1(E_S) + r_1 x^2(E_S) + \lambda r_1 x^3(E_S) \geq r_1\lceil \tfrac{d}{C}\rceil$$

and

$$x^1(E_S) + \min(C, r_2)x^2(E_S) + r_2 x^3(E_S) \geq r_2\lceil \tfrac{d}{\lambda C}\rceil,$$

where $r_1 = r(d, C)$ and $r_2 = r(d, \lambda C)$. The first cut inequality can be strengthened to

$$x^1(E_S) + r_1 x^2(E_S) + \min(\lambda, \lceil \tfrac{d}{C}\rceil)r_1 x^3(E_S) \geq r_1\lceil \tfrac{d}{C}\rceil.$$

Magnanti & Mirchandani [1993] describe a third cut inequality, valid for CS^{UN}, which we will study in Section 6.4.

A new class of cut set inequalities In Section 4.2.2 we found (4.21)

$$cx(E_1) + \big(c - r^Q\big)x(\bar{E}_1) + f^Q(E_1^-) - f^Q(E_1^+) \geq c - r^Q$$

as a new class of cut set inequalities for the polyhedra CS^{BI} and CS^{UN} for the case that $|T| = 1$. We could even prove that it can define a facet if $Q = K^+$. But it was not possible to express it as a *MIR* inequality and hence we cannot apply Proposition 3.14 and lift (4.21) to the multi-facility case. We can only calculate the exact lifting function or find a subadditive upper bound, which will not be done here. For computations and separation we will use (4.21) only for single facility instances (Chapter 7).

4.4 Summary

We have introduced the cut sets CS^{DI}, CS^{BI} and CS^{UN} as relaxations of the network design polyhedra NDP^{DI}, NDP^{BI} and NDP^{UN} in Section 4.1. Cut sets are defined with respect to a node set $S \subset V$ of the underlying networks. It has been shown that facet-defining inequalities for cut sets are facet-defining for the corresponding network design polyhedra if both subgraphs $G[S]$ and $G[V \backslash S]$ are connected (undirected graphs) or strongly connected (directed graphs).

This result has motivated the investigation of the polyhedral structure of cut sets in Section 4.2 for the single facility case. Facet proofs for different classes of so-called flow cut inequalities were provided while emphasising the differences between the three capacity models DIrected, BIdirected, UNdirected. We have been able to generalise well-known inequalities used to solve network design problems.

In Section 4.3 a *MIR* procedure has been developed that produces strong valid inequalities for cut sets in the general multi-commodity, multi-facility case. This procedure is based on lifting facet-defining inequalities of single-facility restrictions by *MIR*.

Chapter 5

Cut sets, upper bounds and flow cover inequalities

5.1 Introduction

In this chapter cut sets with bounded design variables will be investigated. The sets, corresponding to the capacity models DIrected, BIdirected and UNdirected, are

$$
\begin{aligned}
CS^{DI}(u) &= \operatorname{conv}\{\, (f,x) \in CS^{DI} : \ x_a^t \leq u_a^t,\ a \in A_S, t \in T \,\} \\
CS^{BI}(u) &= \operatorname{conv}\{\, (f,x) \in CS^{BI} : \ x_e^t \leq u_e^t,\ e \in E_S, t \in T \,\} \\
CS^{UN}(u) &= \operatorname{conv}\{\, (f,x) \in CS^{UN} : \ x_e^t \leq u_e^t,\ e \in E_S, t \in T \,\}
\end{aligned}
$$

with $u_e^t, u_a^t \in \mathbb{Z}_+ \backslash \{0\}$.

We assume throughout this chapter that the polyhedra are not empty. All demands can be satisfied. For simplicity, additionally suppose that the dimension of the polyhedra is not changing when the bound constraints are added. Hence bounds have to be given large enough.

Since $CS^{DI}(u) \subseteq CS^{DI}$, $CS^{BI}(u) \subseteq CS^{BI}$ and $CS^{UN}(u) \subseteq CS^{UN}$ the cut set inequalities stated in Section 4.2 and Section 4.3 are valid for the bounded cut sets. The *MIR*- procedures given there can still be used. In fact all the results of Chapter 4 hold for $CS^{DI}(u)$, $CS^{BI}(u)$ and $CS^{UN}(u)$ if $u_a^t, u_e^t \geq M$ for all $a \in A_S$, $e \in E_S$, $t \in T$, where M is an integer number large enough.

In order to exploit the special structure of bounded cut sets we will develop strong valid inequalities that are valid only in the presence of bound constraints by simply extending the procedures of Section 4.2 and Section 4.3. In Section 3.3 it was shown that given so-called knapsack sets, bounds on integer variables can be handled by considering covers and packs and by deriving cover inequalities as well as pack inequalities. The analogue for the more complex cut sets or single node flow sets are *flow covers, flow packs* and the corresponding inequalities.

Literature review The polyhedral study of single node flow sets (with bounded design variables) was initiated by Padberg et al. [1985]. They introduce a special case of flow cover inequalities (with only outflow arcs). The generalisation to the inequalities, which will be presented below is from Van Roy & Wolsey [1986].

Important work on the strengthening of flow cover inequalities by superadditive lifting has been carried out by Gu et al. [1999]. Flow pack inequalities were introduced by Stallaert [1997] and investigated in detail by Atamtürk [2001]. Flow pack inequalities are derived as flow cover inequalities after reversing the flow directions.

Atamtürk et al. [2001] state flow cover- and flow pack inequalities for single node flow sets with a very general capacity model. Their results can be used for our cut sets in the multi-facility case.

Louveaux & Wolsey [2003] recently showed how strong valid flow cover- and flow pack inequalities can be obtained by *MIR* and other superadditive lifting functions. The *MIR* procedure they apply to single node flow sets is similar to the one that has been first introduced by Marchand & Wolsey [1998] (see Section 3.4).

A good survey can be found in Wolsey [2003].

Outline of this chapter To derive strong valid flow cover- and flow pack inequalities for $CS^{DI}(u)$, $CS^{BI}(u)$ and $CS^{UN}(u)$ *MIR* procedures are motivated in this chapter that are based on the work of Atamtürk et al. [2001] and Louveaux & Wolsey [2003]. We will especially concentrate on the cut set for directed supply graphs CS^{DI}. A slightly modified *MIR* procedure will then be applied to the sets $CS^{BI}(u)$ and $CS^{UN}(u)$ in Section 5.2.2.

We start with an introduction to flow covers and flow packs for a simple 0-1 single node flow set, which differs from the cut set $CS^{DI}(u)$ in the single-facility case in the way that arc dependent capacities are given. It will be shown how known flow cover- and flow pack inequalities can be derived by a *MIR* procedure based on complementing variables in an appropriate chosen flow cover or flow pack. This procedure is similar to the one given in Section 3.3 for knapsack sets.

Switching to the cut set $CS^{DI}(u)$ we prove that flow cover- and flow pack inequalities reduce to the flow cut inequalities of Section 4.2.1 if only one facility is given (arc independent capacities). It turns out that the procedure of Louveaux & Wolsey [2003] for flow cover- and flow pack inequalities is identical to the procedure that was already developed in Section 4.2.1 if we additionally complement design variables.

A *MIR*-procedure for the general multi-facility case for the the sets $CS^{DI}(u)$, $CS^{BI}(u)$ and $CS^{UN}(u)$ is provided in Section 5.2 to handle bounded design variables and to obtain strong valid flow cover- and flow pack inequalities. If more than one facility is given, the procedure to derive the flow cut inequalities from Section 4.3 will be extended. To obtain flow cover- and flow pack inequalities, variables will be complemented in flow covers and flow packs, defined as in Atamtürk et al. [2001].

First assume that only one facility can be installed on every arc in the dicut A_S. To introduce flow cover- an flow pack inequalities as they are given in the literature, we will first consider a modification of the set $CS^{DI}(u)$ in the single-commodity, single-facility case. Suppose that the capacity to install depends on the arc a.

$$\sum_{a \in A_S^+} f_a - \sum_{a \in A_S^-} f_a \leq d_S \tag{5.1}$$

$$f_a \leq c_a x_a, \quad a \in A_S \tag{5.2}$$

$$x_a \leq u_a, \qquad a \in A_S, \tag{5.3}$$

where $c_a, u_a \in \mathbb{Z}_+\backslash\{0\}$, $d_S \in \mathbb{Z}_+$. The following set the literature refers to as a *single node flow set*.

$$X^{SNF} := \text{conv}\{\,(f,x) \in \mathbb{R}_+^{|A_S|} \times \mathbb{Z}_+^{|A_S|}:\ (f,x) \text{ satisfies } (5.1), (5.2) \text{ and } (5.3)\,\}$$

Under the assumptions $A_S^- \neq \emptyset$ and $d_S + \sum_{a\in A_S^-} c_a u_a \geq u_{\bar{a}}$ for all $\bar{a} \in A_S^-$ or similarly $A_S^- = \emptyset$ and $d_S > 0$, the polyhedron X^{SNF} is full dimensional (see Atamtürk [2001]) in contrast to $CS^{DI}(u)$.

Definition 5.1 *(C^+, C^-) is a **flow cover** for X^{SNF} if*

$$C^+ \subseteq A_S^+, C^- \subseteq A_S^- \quad \textit{and} \quad \sum_{a\in C^+} c_a u_a - \sum_{a\in C^-} c_a u_a - d_S = \lambda > 0.$$

*(P^+, P^-) is a **flow pack** or **reverse flow cover** for X^{SNF} if*

$$P^+ \subseteq A_S^+, P^- \subseteq A_S^- \quad \textit{and} \quad \sum_{a\in P^+} c_a u_a - \sum_{a\in P^-} c_a u_a - d_S = -\mu < 0.$$

In the following results of Louveaux & Wolsey [2003] will be summarised. Strong valid flow cover- and flow pack inequalities for X^{SNF} can be obtained by a certain *MIR* procedure. This will serve as a motivation for a more general procedure for the set $CS^{DI}(u)$.

For simplicity we restrict our attention to the case $u_a = 1\ \forall a \in A_S$. With the same procedure and Definition 5.1, flow cover- and flow pack inequalities are obtained for integer single node flow sets, i. e. $u_a \in \mathbb{Z}_+\backslash\{0\}$ for all $a \in A_S$.

The *MIR* flow cover inequality Suppose (C^+, C^-) is a flow cover for X^{SNF} and choose $\bar{c} \in \mathbb{Z}_+$ with $\bar{c} > \lambda$. Let (C^+, L^+, R^+) and (C^-, L^-, R^-) be partitions of A_S^+ and A_S^-, respectively. The base inequality

$$\sum_{a\in C^+\cup L^+} c_a x_a - \sum_{a\in C^-\cup L^-} c_a x_a - \sum_{a\in R^-} f_a - \sum_{a\in C^+\cup L^+} \bar{f}_a \leq d_S, \tag{5.4}$$

with $\bar{f}_a := c_a x_a - f_a$, is obtained by substituting f_a for $c_a x_a - \bar{f}_a$ for all $a \in C^+ \cup L^+ \cup C^- \cup L^-$ and using the nonnegativity of f_a for $a \in R^+$ and of $\bar{f}_a$ for $a \in C^- \cup L^-$.

Additionally complementing all variables in $C^+ \cup C^-$ gives

$$-\sum_{a\in C^+} c_a \bar{x}_a + \sum_{a\in L^+} c_a x_a + \sum_{a\in C^-} c_a \bar{x}_a - \sum_{a\in L^-} c_a x_a - \sum_{a\in R^-} f_a - \sum_{a\in C^+\cup L^+} \bar{f}_a \leq -\lambda, \tag{5.5}$$

where $\bar{x}_a := 1 - x_a$.

The $\frac{1}{\bar{c}}$-*MIR* inequality for (5.5) is

$$\sum_{a\in C^+} \mathcal{F}_{-\lambda,\bar{c}}(-c_a)\bar{x}_a + \sum_{a\in L^+} \mathcal{F}_{-\lambda,\bar{c}}(c_a) x_a + \sum_{a\in C^-} \mathcal{F}_{-\lambda,\bar{c}}(c_a)\bar{x}_a + \sum_{a\in L^-} \mathcal{F}_{-\lambda,\bar{c}}(-c_a) x_a$$

$$- \sum_{a\in R^-} f_a - \sum_{a\in C^+\cup L^+} \bar{f}_a \leq \mathcal{F}_{-\lambda,\bar{c}}(-\lambda) = -\lambda \tag{5.6}$$

Louveaux & Wolsey [2003] show that if $\bar{c} = \max_{a \in C^+} c_a > \lambda$, then the *MIR* flow cover inequality (5.6) is at least as strong as the $GFC2$ (Generalised Flow Cover) inequality and if $\bar{c} = \max_{a \in C^+ \cup L^-} c_a > \lambda$, (5.6) is at least as strong as the $GFC1$ inequality given by

$$\sum_{a \in C^+} (f_a + (c_a - \lambda)^+(1 - x_a)) - \sum_{a \in L^-} \min(c_a, \lambda) x_a - \sum_{a \in R^-} f_a \leq d + \sum_{a \in C^-} c_a, \qquad (5.7)$$

If $L^+ = \emptyset$ this can easily be seen by using Lemma 3.11 iv). $GFC1$ and $GFC2$ inequalities were introduced by Van Roy & Wolsey [1986].

Proposition 5.2 (Nemhauser & Wolsey [1988]) *The $GFC1$ inequality* (5.7) *defines a facet of X^{SNF} if $C^- = \emptyset$, $\max_{a \in C^+} c_a > \lambda$ and $c_a > \lambda$ for all $a \in L^-$.*

The *MIR* flow pack inequality Suppose (P^+, P^-) is a flow pack for X^{SNF} and choose $\bar{c} \in \mathbb{Z}_+$ with $\bar{c} > \mu$. Let (P^+, L^+, R^+) and (P^-, L^-, R^-) be partitions of A_S^+ and A_S^-, respectively. Aggregating and substituting as for flow cover inequalities and complementing all variables in $P^+ \cup P^-$ gives

$$-\sum_{a \in P^+} c_a \bar{x}_a + \sum_{a \in L^+} c_a x_a + \sum_{a \in P^-} c_a \bar{x}_a - \sum_{a \in L^-} c_a x_a - \sum_{a \in R^-} f_a - \sum_{a \in P^+ \cup L^+} \bar{f}_a \leq \mu,$$

as a valid inequality for X^{SNF}. Now calculating the $\frac{1}{\bar{c}}$-*MIR* inequality yields

$$\sum_{a \in P^+} \mathcal{F}_{\mu,\bar{c}}(-c_a)\bar{x}_a + \sum_{a \in L^+} \mathcal{F}_{\mu,\bar{c}}(c_a) x_a + \sum_{a \in P^-} \mathcal{F}_{\mu,\bar{c}}(c_a)\bar{x}_a + \sum_{a \in L^-} \mathcal{F}_{\mu,\bar{c}}(-c_a) x_a$$
$$- \sum_{a \in R^-} f_a - \sum_{a \in P^+ \cup L^+} \bar{f}_a \leq \mathcal{F}_{\mu,\bar{c}}(\mu) = 0. \qquad (5.8)$$

If $L^- = \emptyset$ and $\bar{c} = \max_{a \in P^- \cup L^+} c_a > \mu$, the *MIR* flow pack inequality (5.8) is as least as strong as the flow pack inequality (reverse flow cover inequality)

$$\sum_{a \in P^+} f_a + \sum_{a \in L^+} (f_a - \min(c_a, \mu) x_a) + \sum_{a \in P^-} (c_a - \mu)^+(1 - x_a) - \sum_{a \in R^-} f_a \leq \sum_{a \in P^+} c_a \qquad (5.9)$$

given in Atamtürk [2001] and Louveaux & Wolsey [2003]. This can be seen by replacing $\mathcal{F}_{\mu,\bar{c}}(-c_a)$ by $-c_a$ for all $a \in P^+$, which is possible since $0 \geq \mathcal{F}_{\mu,\bar{c}}(-c_a) \geq -c_a$ (Corollary 3.8) and using Lemma 3.11 iv).

Flow pack inequalities may be facet-defining for the restriction

$$X_{P^+}^{SNF} = \text{conv}\{\, (f, x) \in X^{SNF} : \ x_a = 1 \ \forall a \in P^+ \,\}$$

of X^{SNF} obtained by fixing all variables in P^+ to their upper bound. In this context (5.8) is obtained by lifting (5.9) to a valid inequality for X^{SNF} using the superadditive function $\mathcal{F}_{\mu,\bar{c}}$. A weakened formulation of a result of Atamtürk [2001] is:

Proposition 5.3 (Atamtürk [2001]) (5.9) *defines a facet of $X_{P^+}^{SNF}$ if $\max_{a \in P^-} c_a > \mu$, $c_a > \mu$ for all $a \in L^+$ and $R^- \cup P^+ \neq \emptyset$.*

Remark 5.4 *The Propositions 5.2 and 5.3 indicate that in order to derive strong valid flow cover- and flow pack inequalities, it is necessary to choose flow covers and flow packs such that λ or μ are small. This is true for all kinds of single node flow sets (see Gu et al. [1999], Atamtürk [2001], Atamtürk et al. [2001] and Louveaux & Wolsey [2003]).*

*The excess λ for flow covers or the residual μ for flow packs should be smaller than certain coefficients of the base inequality. The same holds for all the kinds of knapsack sets and the corresponding covers and packs (see Atamtürk [2003*a*] for a survey).*

Note that a flow cover for X^{SNF} can be seen as a cover with respect to the single constraint (5.4)*. Similarly, a flow pack is a pack with respect to the inequality that is obtained by aggregating and substituting. We introduced covers and packs for mixed knapsack sets in Section 3.3.*

Flow cover inequalities and CS^{DI} We reviewed that strong flow cover- and flow pack inequalities can be obtained with a certain *MIR* procedure introduced by Marchand & Wolsey [1998] and applied to single node flow sets by Louveaux & Wolsey [2003]. The procedure consists of the five steps **Aggregating**, **Substituting**, **Complementing**, **Scaling** and ***MIR*** .

In the following we will show that the procedure is in fact just an extension to the one that was developed to derive the strong valid flow cut inequalities for CS^{DI} in Section 4.2.1. Set

$$Y^{SNF} := \text{conv}\{\, (f,x) \in X^{SNF} : \; f(A_S^+) - f(A_S^-) = d_S \,\}.$$

Hence Y^{SNF} is $CS^{DI}(u)$ but with arc dependent capacities. Note that now we can reverse the directions of the flow and consider a relaxation of the flow conservation constraint for Y^{SNF} of the form

$$f(A_S^-) - f(A_S^+) \leq -d_S.$$

Thus flow packs for Y^{SNF} (or $CS^{DI}(u)$) are flow overs when multiplying the flow conservation constraint by -1. In terms of the underlying graph, switching between flow covers and flow packs means switching between the cut set for S and the cut set for $V\backslash S$. So, in the sequel we can concentrate on flow covers and flow cover inequalities.

Let (C^+, L^+, R^+) and (C^-, L^-, R^-) be partitions of A_S^+ and A_S^-, respectively, where (C^+, C^-) is a flow cover, as defined above. Set $A_1^+ := C^+ \cup L^+$ and $A_2^- := C^- \cup L^-$. Aggregating and substituting as in Section 4.2.1 gives the base inequality

$$\sum_{a \in A_1^+} c_a x_a - \sum_{a \in A_2^-} c_a x_a + f(\bar{A}_1^+) + \bar{f}(C^- \cup L^-) \geq d_S.$$

Notice that $\bar{A}_1^+ = R^+$. Extending the procedure from Section 4.2.1 by additionally complementing variables in $C^+ \cup C$ yields

$$-\sum_{a \in C^+} c_a \bar{x}_a + \sum_{a \in L^+} c_a x_a + \sum_{a \in C^-} c_a \bar{x}_a - \sum_{a \in L^-} c_a x_a + f(\bar{A}_1^+) + \bar{f}(C^- \cup L^-) \geq -\lambda. \tag{5.10}$$

Choose $\bar{c} > \lambda$. The $\frac{1}{\bar{c}}$-*MIR* inequality for (5.10) is

$$\sum_{a \in C^+} \mathcal{G}_{-\lambda,\bar{c}}(-c_a)\bar{x}_a + \sum_{a \in L^+} \mathcal{G}_{-\lambda,\bar{c}}(c_a)x_a + \sum_{a \in C^-} \mathcal{G}_{-\lambda,\bar{c}}(c_a)\bar{x}_a + \sum_{a \in L^-} \mathcal{G}_{-\lambda,\bar{c}}(-c_a)x_a + f(R^+) + \bar{f}(C^- \cup L^-) \geq 0. \tag{5.11}$$

In the following we show that inequality (5.11) is equivalent to the *MIR*-flow cover inequality (5.6) up to a scalar multiple of the flow conservation constraint $f(A_S^+) - f(A_S^-) = d_S$. Hence both inequalities induce the same face of Y^{SNF}.

Lemma 5.5 *If* $\lambda < \bar{c}$ *then* (5.11) *and* (5.6) *induce the same face of* Y^{SNF}.

Proof. Adding to (5.11) the flow conservation constraint $f(A_S^-) - f(A_S^+) = -d_S$ gives

$$\sum_{a\in C^+} \mathcal{G}_{-\lambda,\bar{c}}(-c_a)\bar{x}_a + \sum_{a\in L^+} \mathcal{G}_{-\lambda,\bar{c}}(c_a)x_a + \sum_{a\in C^-} \mathcal{G}_{-\lambda,\bar{c}}(c_a)\bar{x}_a + \sum_{a\in L^-} \mathcal{G}_{-\lambda,\bar{c}}(-c_a)x_a$$
$$- f(C^+ \cup L^+) + \sum_{a\in C^-\cup L^-} c_a x_a + f(R^-) \geq -d_S$$

with $\mathcal{G}_{-\lambda,\bar{c}}(c_a) = -\mathcal{F}_{\lambda,\bar{c}}(-c_a)$ reducing to

$$\sum_{a\in C^+} \mathcal{F}_{\lambda,\bar{c}}(c_a)\bar{x}_a + \sum_{a\in L^+} \mathcal{F}_{\lambda,\bar{c}}(-c_a)x_a + \sum_{a\in C^-} \mathcal{F}_{\lambda,\bar{c}}(-c_a)\bar{x}_a + \sum_{a\in L^-} \mathcal{F}_{\lambda,\bar{c}}(c_a)x_a$$
$$+ f(C^+ \cup L^+) - \sum_{a\in C^-\cup L^-} c_a x_a - f(R^-) \leq d_S.$$

Using that $\mathcal{F}_{\lambda,\bar{c}}(-c_a) = \mathcal{F}_{-\lambda,\bar{c}}(c_a) - c_a$ if $\langle \frac{\bar{c}}{\lambda} \rangle > 0$ (Lemma 3.11 iii)) yields

$$\sum_{a\in C^+} \mathcal{F}_{-\lambda,\bar{c}}(-c_a)\bar{x}_a + \sum_{a\in L^+} \mathcal{F}_{-\lambda,\bar{c}}(c_a)x_a + \sum_{a\in C^-} \mathcal{F}_{-\lambda,\bar{c}}(c_a)\bar{x}_a + \sum_{a\in L^-} \mathcal{F}_{-\lambda,\bar{c}}(-c_a)x_a$$
$$- \bar{f}(C^+ \cup L^+) - f(R^-) \leq -\lambda,$$

which is inequality (5.6). We have shown that (5.11) is (5.6) up to a scalar multiple of the flow conservation constraint. Hence both inequalities induce the same face of Y^{SNF}. ∎

It turns out that in presence of the flow conservation constraint $f(A_S^+) - f(A_S^-) = d_S$ the *MIR* procedures of Section 4.2 and Section 4.3, augmented with complementing, can be used to obtain flow cover inequalities. Hence flow cover inequalities generalise flow cut inequalities.

The single facility case For the set $CS^{DI}(u)$ in the single-facility case it will be shown now that the extended procedure does not provide a new class of inequalities. A capacity c can be installed independent from $a \in A_S$. The conditions from Proposition 5.2 and Proposition 5.3 concerning the size of λ and μ reduce to $\lambda < c$ and $\mu < c$.

Lemma 5.6 *If* $c_a = c$ *for all* $a \in A_S$ *and* $\lambda < c$*, the* MIR *flow cover inequality reduces to the flow cut inequality* (4.14) *of Section 4.2.1.*

Proof. Let (C^+, C^-) be a flow cover. Hence

$$\sum_{a\in C^+} cu_a - \sum_{a\in C^-} cu_a - d_S = c(u(C^+) - u(C^-)) - d_S = \lambda > 0.$$

With $(u(C^+) - u(C^-)) \in \mathbb{Z}$ and $c > \lambda$ follows that $\langle \frac{d_S}{c} \rangle > 0$, $(u(C^+) - u(C^-)) = \lceil \frac{d_S}{c} \rceil$ and $\lambda = c - r(d_S, c) =: c - r$. But then

$$\mathcal{G}_{-\lambda,\bar{c}}(c_a) = \mathcal{G}_{r-c,c}(c) = r \quad \text{and} \quad \mathcal{G}_{-\lambda,\bar{c}}(-c_a) = \mathcal{G}_{r-c,c}(-c) = -r.$$

and inequality (5.11) takes the form

$$-\sum_{a\in C^+} r\bar{x}_a + \sum_{a\in L^+} rx_a + \sum_{a\in C^-} r\bar{x}_a - \sum_{a\in L^-} rx_a + \sum_{a\in R^+} f_a + \sum_{a\in C^-\cup L^-} \bar{f}_a \geq 0$$

$$\Longleftrightarrow$$

$$rx(C^+\cup L^+) - rx(C^-\cup L^-) + f(R^+) + \bar{f}(C^-\cup L^-) \geq r\lceil \tfrac{d_S}{c}\rceil,$$

which after setting $A_1^+ := C^+ \cup L^+$ and $A_2^- := C^- \cup L^-$ leads to the flow cut inequality (4.14). ∎

It was shown that for single-facility bounded cut sets with arc independent capacities we cannot obtain a new class of strong valid inequalities with the *MIR* procedure described above. Complementing does not change the corresponding *MIR* inequalities. The question that arises is, how can the special structure of such sets be exploited? The author conjectures that the bound constraints itself suffice here. The sets $CS^{DI}(u)$, CS^{BI} and $CS^{UN}(u)$ are completely described by all cut set inequalities from Section 4.2 together with all bound constraints.

Example 5.7 *Consider the cut sets from Example 4.5 and Example 4.16 but now with bounded design variables:*

$$\begin{aligned} CS^{DI}(u) = \operatorname{conv}\{x\in\mathbb{Z}^4, f\in\mathbb{R}^4 \mid\ & f_1 + f_2 - f_3 - f_4 = 7 \\ & 0 \leq f_i \leq 3x_i,\ \ i\in\{1,2,3,4\} \\ & x_i \leq u_i\} \end{aligned}$$

with $u = (1,3,2,1)$ *and*

$$\begin{aligned} CS^{BI}(u) = \operatorname{conv}\{x\in\mathbb{Z}^2, f\in\mathbb{R}^4 \mid\ & f_1 + f_2 - f_3 - f_4 = 7, \\ & 0 \leq f_i \leq 3x_1 \leq 6,\ \ i\in\{1,3\}, \\ & 0 \leq f_i \leq 3x_2 \leq 9,\ \ i\in\{2,4\}\}, \end{aligned}$$

$$\begin{aligned} CS^{UN}(u) = \operatorname{conv}\{x\in\mathbb{Z}^2, f\in\mathbb{R}^4 \mid\ & f_1 + f_2 - f_3 - f_4 = 7, \\ & f_1 + f_3 \leq 3x_1 \leq 6, \\ & f_2 + f_4 \leq 3x_2 \leq 9, \\ & 0 \leq f_i,\ \ i\in\{1,\ldots,4\}\}. \end{aligned}$$

Adding all flow cut inequalities stated in Example 4.5 to the linear relaxation of $CS^{DI}(u)$ *yields a complete description of* $CS^{DI}(u)$ *(PORTA, Christof & Löbel [2005]). The same holds for* $CS^{BI}(u)$ *as well as* $CS^{UN}(u)$ *and the cut set inequalities given in Example 4.16 (PORTA, Christof & Löbel [2005]).*

Summary We gave an introduction to flow covers and flow packs and showed that flow cover inequalities generalise flow cut inequalities for CS^{DI}. They are obtained by the same *MIR* procedure but by additional complementing variables in the flow cover. In the single-facility case this procedure does not provide a new class of inequalities.

5.2 A *MIR* procedure in the multi-facility case

In this section $CS^{DI}(u)$ is considered to be given with a set of facilities T having a cardinality of at least 2. We first restrict our attention to cut sets with DIrected capacity constraints. The *MIR* procedure that is going to be developed can be applied to $CS^{BI}(u)$ and $CS^{UN}(u)$ in a similar way.

5.2.1 DIrected capacity constraints

If there is more than one commodity to route we choose a subset Q of the commodities and aggregate as already shown in Section 4.2 to arrive at a system of the form

$$\begin{aligned} f^Q(A_S^+) - f^Q(A_S^-) &= d_S^Q \\ f_a^Q &\le \sum_{t \in T} c^t x_a^t \qquad \forall a \in A_S \\ 0 &\le x_a^t \qquad\quad \forall a \in A_S, t \in T \end{aligned}$$

In the last section it was shown that to obtain strong valid flow cover inequalities we simply have to extend the *MIR* procedure of Section 4.2. This approach will be used in the multi-facility case too. We will apply the steps **aggregating** and **substituting** to arrive at the base inequality (4.25) that was used in Section 4.3.1 to obtain the flow cut inequalities (4.27). But before **scaling and *MIR*** we will **complement** design variables in previously chosen flow covers. It turns out that in contrast to the single-facility case this extended procedure provides new classes of strong valid inequalities.

To be able to calculate flow cover inequalities in the multi-facility case, the definition of flow covers has to be extended.

Definition 5.8 (C^+, C^-) *is a* ***flow cover*** *for* $CS^{DI}(u)$ *if*

$$C^+ \subseteq A_S^+, C^- \subseteq A_S^- \quad \text{and} \quad \sum_{a \in C^+, t \in T} c^t u_a^t - \sum_{a \in C^-, t \in T} c^t u_a^t - d_S^Q = \lambda > 0.$$

Note again that considering the cut set for $V \backslash S$ by multiplying the flow conservation constraint $f(A_S^+) - f(A_S^-) = d_S^Q$ by -1 we obtain reverse flow covers for $CS^{DI}(u)$ with the same definition. Hence we can concentrate on flow covers.

If $u_a^t = 1$ for all $a \in A_S, t \in T$, then this definition can be seen as a special case of the definition first given by Atamtürk et al. [2001], who investigate the most general single node flow set given in the literature. Their additive variable upper bound capacity constraints, in fact, generalise the DIrected and even the BIdirected case considered in this thesis.

Before we explicitly state the *MIR* flow cover inequalities, a little example is given to demonstrate our approach.

Example 5.9 *Here we consider the cut set from Example 4.35 given in Section 4.3 but with bounds on the design variables:*

$$\begin{aligned} CS^{DI}(1) = \text{conv}\{x \in \mathbb{Z}_+^6, f \in \mathbb{R}_+^3 \mid\ & f_1 + f_2 - f_3 = 3 \\ & f_i \le 2x_i^1 + 5x_i^2, \\ & x_i \le 1, \quad i \in \{1,2,3\}\} \end{aligned}$$

$CS^{DI}(1)$ has dimension 8. The flow cut inequalities stated in Example 4.35 are all valid for $CS^{DI}(1)$ since $CS^{DI}(1) \subset CS^{DI}$. From the facet-defining flow cut inequalities for CS^{DI} (4.30b), (4.30d), (4.30f), (4.30h) *and* (4.30l) *still define facets of $CS^{DI}(1)$ while* (4.30a2), (4.30e2) *and* (4.30i2) *are faces of dimension 6. The remaining flow cut inequalities are weak for $CS^{DI}(1)$ (inducing faces of dimension 3 to 5).*

There are four flow covers with respect to Definition 5.8:

$$\begin{aligned}(C^+, C^-) =& (\{1\}, \emptyset) && \lambda = 4,\\ (C^+, C^-) =& (\{2\}, \emptyset) && \lambda = 4,\\ (C^+, C^-) =& (\{1,2\}, \emptyset) && \lambda = 11,\\ (C^+, C^-) =& (\{1,2\}, \{3\}) && \lambda = 4.\end{aligned}$$

Choosing the cover $(C^+, C^-) = (\{1\}, \{\emptyset\})$ it will be shown in the following how to derive a flow cover inequality. C^+ has to be a subset of A_1^+ and C^- a subset of A_2^-. Setting $A_1^+ = \{1\}$ and $A_2^- = \{3\}$ the base inequality is

$$f_2 + \bar{f}_3 + 2x_1^1 + 5x_1^2 - 2x_3^1 - 5x_3^2 \geq 3$$

where $\bar{f}_3 = 2x_3^1 + 5x_3^2 - f_3$. Note that with respect to this inequality $(C^+ \cup C^-)$ is a cover as defined in Section 3.3. Complementing variables in the flow cover yields

$$f_2 + \bar{f}_3 - 2\bar{x}_1^1 - 5\bar{x}_1^2 - 2x_3^1 - 5x_3^2 \geq -4.$$

*Setting $\bar{c} = 5 > \lambda$ and calculating the $\frac{1}{\bar{c}}$-*MIR *inequality gives*

$$\begin{aligned} f_2 + \bar{f}_3 - \bar{x}_1^2 - x_3^2 &\geq 0 \quad \Longleftrightarrow\\ f_2 + 2x_3^1 + 4x_3^2 - f_3 + x_1^2 &\geq 1\end{aligned}$$

defining a facet of $CS^{DI}(1)$. We will now state all possible flow cover inequalities with respect to Definition 5.8:

$$f_2 + x_1^2 \geq 1 \quad (C^+ = \{1\}, C^- = \emptyset, A_1^+ = \{1\}, A_2^- = \emptyset) \tag{5.12a}$$

$$f_2 + 2x_3^1 + 4x_3^2 - f_3 + x_1^2 \geq 1 \quad (C^+ = \{1\}, C^- = \emptyset, A_1^+ = \{1\}, A_2^- = \{3\}) \tag{5.12b}$$

$$x_1^2 + x_2^1 + x_2^2 \geq 1 \quad (C^+ = \{1\}, C^- = \emptyset, A_1^+ = \{1,2\}, A_2^- = \emptyset) \tag{5.12c}$$

$$2x_3^1 + 4x_3^2 - f_3 + x_1^2 + x_2^1 + x_2^2 \geq 1 \quad (C^+ = \{1\}, C^- = \emptyset, A_1^+ = \{1,2\}, A_2^- = \{3\}) \tag{5.12d}$$

$$f_1 + x_2^2 \geq 1 \quad (C^+ = \{2\}, C^- = \emptyset, A_1^+ = \{2\}, A_2^- = \emptyset) \tag{5.12e}$$

$$f_1 + 2x_3^1 + 4x_3^2 - f_3 + x_2^2 \geq 1 \quad (C^+ = \{2\}, C^- = \emptyset, A_1^+ = \{2\}, A_2^- = \{3\}) \tag{5.12f}$$

$$x_1^1 + x_1^2 + x_2^2 \geq 1 \quad (C^+ = \{2\}, C^- = \emptyset, A_1^+ = \{1,2\}, A_2^- = \emptyset) \tag{5.12g}$$

$$2x_3^1 + 4x_3^2 - f_3 + x_1^1 + x_1^2 + x_2^2 \geq 1 \quad (C^+ = \{2\}, C^- = \emptyset, A_1^+ = \{1,2\}, A_2^- = \{3\}) \tag{5.12h}$$

$$0 \geq 0 \quad (C^+ = \{1,2\}, C^- = \emptyset, A_1^+ = \{1,2\}, A_2^- = \emptyset) \tag{5.12i}$$

$$2x_3^1 + 5x_3^2 - f_3 \geq 0 \quad (C^+ = \{1,2\}, C^- = \emptyset, A_1^+ = \{1,2\}, A_2^- = \{3\}) \tag{5.12j}$$

$$x_3^1 + 4x_3^2 - f_3 + x_1^2 + x_2^2 \geq 0 \quad (C^+ = \{1,2\}, C^- = \{3\}, A_1^+ = \{1,2\}, A_2^- = \{3\}) \tag{5.12k}$$

All inequalities, based on flow covers with excess of $\lambda = 4$*, are not trivial and are not equivalent to already known cut set inequalities. All define facets for* $CS^{DI}(1)$*. Note that* $4 < 5 = \max(c_t)_{t\in T}$*, thus* λ *is small with respect to the coefficients of the base inequalities.*

Both (5.12i) *and* (5.12j)*, based on the flow cover* $(C^+, C^-) = (\{1,2\}, \emptyset)$ *with large excess* $\lambda = 11$*, are trivial.*

Reversing the flow directions we could consider the flow conservation constraint $f_3 - f_1 + f_2 = -3$ *in order to obtain reverse flow covers. But since* $\sum_{t\in T} c^t u_a^t = 7 > 3$ *for all* $a \in \{1,2,3\}$*, there is no reverse flow cover for* $CS^{DI}(1)$

From the last example we draw the following conclusions:

- Extending the *MIR* procedure of Section 4.3 by additionally complementing variables in a flow cover provides a new class of strong valid inequalities for the set $CS^{DI}(u)$.
- A shortcoming of Definition 5.8 is that it is not very general. It may happen that there are no flow covers or that all flow covers have large excess λ.

The second point can be handled by generalising the definition of a flow cover. We follow the approach of Atamtürk et al. [2001]. We will (implicitly) consider restrictions of $CS^{DI}(u)$ by fixing design variables to their lower bound zero. Flow covers will be defined with respect to such restrictions. The corresponding flow cover inequalities can be seen as lifted inequalities. The lifting from valid flow cover inequalities of the restrictions to valid inequalities of $CS^{DI}(u)$ is done by using the same subadditive *MIR* function that was used to generate the (restricted) flow cover inequalities, such that we can explicitly state the lifted inequalities without considering neither the restrictions nor the restricted inequalities (see Proposition 3.14 about superadditive (subadditive) lifting with *MIR*).

Note that this approach can be extended by using other, possibly stronger valid superadditive (subadditive) lifting functions to lift the *MIR* flow cover inequalities, as it has been done by Atamtürk et al. [2001] and by Louveaux & Wolsey [2003]. In this thesis we solely consider *MIR* as a lifting procedure.

Definition 5.10 *Let* $T_a \subseteq T$*,* $T_a \neq \emptyset$ *for all* $a \in A_S$*.*
(C^+, C^-) *is a* ***generalised flow cover*** *for* $CS^{DI}(u)$ *with respect to the sets* $T_a, a \in A_S$ *if*

$$C^+ \subseteq A_S^+, C^- \subseteq A_S^- \quad \text{and} \quad \sum_{a\in C^+}\sum_{t\in T_a} c_a^t u_a^t - \sum_{a\in C^-}\sum_{t\in T_a} c_a^t u_a^t - d_S^Q = \lambda > 0.$$

Notice that setting $T_a = T$ for all $a \in A_S$ gives Definition 5.8. Again, if $u_a^t = 1$ for all $a \in A_S, t \in T$, then this definition is a special case of the definition of generalised flow covers of Atamtürk et al. [2001].

Example 5.9 (continued) *With the extended definition there is a bunch of generalised flow covers for the set* $CS^{DI}(1)$*. We only state two examples. Set* $T_1 = \{1\}$ *and* $T_2 = \{1\}$*. It follows that* $(C^+, C^-) = (\{1,2\}, \emptyset)$ *is a generalised flow cover with excess* $\lambda = 1$*. Taking the base inequality*

$$\bar{f}_3 + 2x_1^1 + 5x_1^2 + 2x_2^1 + 5x_2^2 - 2x_3^1 - 5x_3^2 \geq 3,$$

complementing x_1^1 *and* x_2^1 *and calculating the* $\frac{1}{5}$-MIR *inequality gives*

$$\begin{aligned} \bar{f}_3 - \bar{x}_1^1 + 4x_1^2 - \bar{x}_2^1 + 4x_2^2 - x_3^1 - 4x_3^2 &\geq 0 \iff \\ f_2 + x_3^1 + x_3^2 - f_3 + x_1^1 + 4x_1^2 + x_2^1 + 4x_2^2 &\geq 2 \end{aligned}$$

defining a facet of $CS^{DI}(1)$.

Now set $T_1 = \{1,2\}$, $T_2 = \{2\}$ *and* $T_3 = \{1,2\}$. *Hence* $(C^+, C^-) = (\{1,2\},\{3\})$ *is a generalised flow cover with excess* $\lambda = 2$. *Taking the same base inequality, complementing in* x_1^1, x_1^2, x_2^2, x_3^1, x_3^2 *and calculating the* $\frac{1}{5}$-MIR *inequality results in*

$$\begin{aligned} \bar{f}_3 - 3\bar{x}_1^2 + 2x_2^1 - 3\bar{x}_2^2 + 2\bar{x}_3^1 + 3\bar{x}_3^2 &\geq 0 \iff \\ 2x_3^2 - f_3 + 3x_1^2 + 2x_2^1 + 3x_2^2 &\geq 1, \end{aligned}$$

which is facet-defining.

We will now generalise the last example and will describe the procedure to obtain flow cover inequalities for the set $CS^{DI}(u)$ in the multi-facility case.

Aggregating and Substituting Aggregate and Substitute as in Section 4.3.1 to arrive at the base inequality

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + \sum_{t \in T} c^t \big(x^t(A_1^+) - x^t(A_2^-)\big) \geq d_S^Q,$$

where $A_1^+ \subseteq A_S^+$, $A_2^- \subseteq A_S^-$ and Q a subset of the commodities K. The arc sets A_1^+, A_2^- and the commodity set Q are chosen as described in Section 4.3.1.

Complementing This step is an extension to the procedure in Section 4.3.1. Choose sets $T_a \subseteq T$, $T_a \neq \emptyset$ for all $a \in A_S$ and a generalised flow cover (C^+, C^-) with small excess λ such that $C^+ \subseteq A_1^+$ and $C^- \subseteq A_2^-$. We restrict our attention to flow covers with $\lambda < \max(c^t)_{t \in T}$. Let $L^+ := A_1^+ \backslash C^+$, $L^- := A_2^- \backslash C^-$ and $\bar{T}_a := T \backslash T_a$ for $a \in A_S$. Complementing all design variables in the chosen flow cover yields

$$\begin{aligned} f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) - \sum_{a \in C^+, t \in T_a} c^t \bar{x}_a^t + \sum_{a \in C^+, t \in \bar{T}_a} c^t x_a^t + \sum_{a \in L^+, t \in T} c^t x_a^t \\ + \sum_{a \in C^-, t \in T_a} c^t \bar{x}_a^t - \sum_{a \in C^-, t \in \bar{T}_a} c^t x_a^t - \sum_{a \in L^-, t \in T} c^t x_a^t \geq -\lambda. \end{aligned}$$

Scaling and *MIR* Set $\bar{c} := \max(c^t)_{t \in T}$. Hence $\bar{c} > \lambda$ and $\bar{c} \geq c^t$ for all $t \in T$. Now calculate the $\frac{1}{\bar{c}}$-*MIR* inequality

$$\begin{aligned} & f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) \\ & \quad + \sum_{a \in C^+, t \in T_a} \mathcal{G}_{-\lambda,\bar{c}}(-c^t)\bar{x}_a^t + \sum_{a \in C^+, t \in \bar{T}_a} \mathcal{G}_{-\lambda,\bar{c}}(c^t)x_a^t + \sum_{a \in L^+, t \in T} \mathcal{G}_{-\lambda,\bar{c}}(c^t)x_a^t \\ & \quad + \sum_{a \in C^-, t \in T_a} \mathcal{G}_{-\lambda,\bar{c}}(c^t)\bar{x}_a^t + \sum_{a \in C^-, t \in \bar{T}_a} \mathcal{G}_{-\lambda,\bar{c}}(-c^t)x_a^t + \sum_{a \in L^-, t \in T} \mathcal{G}_{-\lambda,\bar{c}}(-c^t)x_a^t \geq 0. \end{aligned}$$

Using Lemma 3.11 iv) leads to

$$
\begin{aligned}
&f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) \\
&\qquad - \sum_{a\in C^+, t\in T_a} (c_a-\lambda)^+ \bar{x}_a^t + \sum_{a\in C^+, t\in \bar{T}_a} \min(c_a, \bar{c}-\lambda) x_a^t + \sum_{a\in L^+, t\in T} \min(c_a, \bar{c}-\lambda) x_a^t \\
&\qquad + \sum_{a\in C^-, t\in T_a} \min(c_a, \bar{c}-\lambda) \bar{x}_a^t - \sum_{a\in C^-, t\in \bar{T}_a} (c_a-\lambda)^+ x_a^t - \sum_{a\in L^-, t\in T} (c_a-\lambda)^+ x_a^t \geq 0.
\end{aligned}
$$

Note that the last inequality is still valid if $\bar{c} > \max(c^t)_{t\in T}$. But it gets weaker. $\bar{c} = \max(c^t)_{t\in T}$ is the best choice here. Rewriting in the space of original variables gives

$$
\begin{aligned}
&f^Q(\bar{A}_1^+) - f^Q(A_2^-) \\
&\qquad + \sum_{a\in C^+, t\in T_a} (c_a-\lambda)^+ x_a^t + \sum_{a\in C^+, t\in \bar{T}_a} \min(c_a, \bar{c}-\lambda) x_a^t + \sum_{a\in L^+, t\in T} \min(c_a, \bar{c}-\lambda) x_a^t \\
&\qquad + \sum_{a\in C^-, t\in T_a} (c_a-\bar{c}+\lambda)^+ x_a^t + \sum_{a\in C^-, t\in \bar{T}_a} \min(c_a, \lambda) x_a^t + \sum_{a\in L^-, t\in T} \min(c_a, \lambda) x_a^t \\
&\qquad \geq \sum_{a\in C^+, t\in T_a} (c_a-\lambda)^+ u_a^t - \sum_{a\in C^-, t\in T_a} \min(c_a, \bar{c}-\lambda) u_a^t, \qquad (5.13)
\end{aligned}
$$

which we call a (generalised) *MIR* flow cover inequality for the polytope $CS^{DI}(u)$ in the multi-facility case.

Similar to Lemma 5.5 it can be shown that by adding a scalar multiple of the flow conservation constraint inequality (5.13) reduces to the additive $\leq$-*MIR* flow cover inequality stated in Louveaux & Wolsey [2003].

Suppose $T_a = T$ for all $a \in C^+$ and $C^- \cup L^+ = \emptyset$. If additionally $\max(c^t)_{t\in T} > \lambda$ then (5.13) reduces to the additive flow cover inequality of Atamtürk et al. [2001], which is facet-defining for the set considered there under these conditions (Atamtürk et al. [2001, Proposition 3]). This generalises Proposition 5.2 and provides the still missing motivation to only consider flow covers with $\max(c^t)_{t\in T} > \lambda$.

Flow cover inequalities may be strong and non-redundant under weaker conditions (Atamtürk et al. [2001, Remark 2]) as for instance inequality (5.12k), which is facet-defining although $C^- \neq \emptyset$.

If $T_a \neq T$ for some $a \in C^+ \cup C^-$, then (5.13) can be seen as obtained by lifting variables in $\bar{T}_a, a \in C^+ \cup C^-$ using the valid subadditive *MIR* function $\mathcal{G}_{-\lambda,\bar{c}}$. For other valid lifting functions see Atamtürk et al. [2001] and Louveaux & Wolsey [2003].

Summary Based on the observations of Section 5.1 we developed a *MIR* procedure that can be used to separate flow cover inequalities for $CS^{DI}(u)$ in the general multi-commodity multi-facility case. It turned out that this *MIR* procedure provides a large class of strong valid inequalities different from flow cut inequalities.

5.2.2 BIdirected and UNdirected capacity constraints

In this section we consider the sets $CS^{BI}(u)$ and $CS^{UN}(u)$ in the multi-facility case. We will simply apply the *MIR* procedure developed in the last section.

In Section 4.3.2 we stated valid base inequalities for the sets $CS^{BI}(u)$ and $CS^{UN}(u)$ that were used to obtain the flow cut inequalities of type (4.33) by *MIR*.

The same base inequalities will be used here but before applying the final *MIR* step, variables will be complemented in a previously chosen flow cover.

Definition 5.11 *Let $T_e \subseteq T$, $T_e \neq \emptyset$ for all $e \in E_S$.*
(C_1, C_2) is a ***generalised flow cover*** *for $CS^{BI}(u)$ and $CS^{UN}(u)$ with respect to the sets $T_e, e \in E_S$ if*

$$C_1, C_2 \subseteq E_S \quad \textit{and} \quad \sum_{e \in C_1} \sum_{t \in T_e} c_e^t u_e^t - \sum_{e \in C_2} \sum_{t \in T_e} c_e^t u_e^t - d_S^Q = \lambda > 0.$$

Note that not necessarily $C_1 \cap C_2 = \emptyset$. We will now describe the *MIR* procedure that produces strong valid flow cover inequalities for the sets $CS^{BI}(u)$ and $CS^{UN}(u)$.

Aggregating and Substituting In Section 4.3.2 it was shown how to obtain (4.31)

$$f^Q(\bar{E}_1^+) + \bar{f}^Q(E_2^-) + \sum_{t \in T} c^t \big(x^t(E_1) - x^t(E_2)\big) \geq d_S^Q,$$

as a valid base inequality for the sets CS^{BI} and CS^{UN}. Since $CS^{BI}(u) \subseteq CS^{BI}$ and $CS^{UN}(u) \subseteq CS^{UN}$, this inequality is also valid for the bounded cut sets. Remember that

$$\bar{f}^Q(E_2^-) = \sum_{t \in T} c^t x^t(E_2) - f^Q(E_2^-) \geq 0$$

and E_1, E_2 are subsets of the cut E_S. We will use inequality (4.31) as a base inequality for the derivation of flow cover inequalities. We choose the sets Q, E_1 and E_2 with the same restrictions as in (4.3.2). So for the BIdirected bounded cut sets $CS^{BI}(u)$ we only consider the case $E_1 \backslash E_2 \neq \emptyset$ and for $CS^{UN}(u)$ we restrict ourselves to $E_1 \cap E_2 = \emptyset$ (see Lemma 4.17).

Complementing, Scaling and *MIR* We start with the base inequality (4.31). As an extension to the procedure in Section 4.3.2 choose sets $T_e \subseteq T$, $T_e \neq \emptyset$ for all $e \in E_S$ and a generalised flow cover (C_1, C_2) with excess $\lambda < \max(c^t)_{t \in T}$ such that $C_1 \subseteq E_1$ and $C_2 \subseteq E_2$.

Notice that if $C_1 \cup C_2 \subseteq E_1 \cap E_2$ there are no variables to complement in the base inequality (4.31). Suppose $(C_1 \cup C_2) \backslash (E_1 \cap E_2) \neq \emptyset$.

Let $L_1 := E_1 \backslash C_1$, $L_2 := E_2 \backslash C_2$ and $\bar{T}_e := T \backslash T_e$ for $e \in E_S$. Complementing all design variables in the chosen flow cover yields

$$\begin{aligned} f^Q(\bar{E}_1^+) + \bar{f}^Q(E_2^-) &- \sum_{a \in C_1, t \in T_e} c^t \bar{x}_e^t + \sum_{e \in C_1, t \in \bar{T}_e} c^t x_e^t + \sum_{e \in L_1, t \in T} c^t x_e^t \\ &+ \sum_{e \in C_2, t \in T_e} c^t \bar{x}_e^t - \sum_{e \in C_2, t \in \bar{T}_e} c^t x_e^t - \sum_{e \in L_2, t \in T} c^t x_e^t \geq -\lambda. \end{aligned}$$

Similar to the last section we get

$$\begin{aligned} & f^Q(\bar{E}_1^+) - f^Q(E_2^-) \\ & \quad + \sum_{e\in C_1, t\in T_e} (c_e-\lambda)^+ x_e^t + \sum_{e\in C_1, t\in \bar{T}_e} \min(c_e, \bar{c}-\lambda) x_e^t + \sum_{e\in L_1, t\in T} \min(c_e, \bar{c}-\lambda) x_e^t \\ & \quad + \sum_{e\in C_2, t\in T_e} (c_e-\bar{c}+\lambda)^+ x_e^t + \sum_{e\in C_2, t\in \bar{T}_e} \min(c_e, \lambda) x_e^t + \sum_{e\in L_2, t\in T} \min(c_e, \lambda) x_e^t \\ & \quad \geq \sum_{e\in C_1, t\in T_e} (c_e-\lambda)^+ u_e^t - \sum_{e\in C_2, t\in T_e} \min(c_e, \bar{c}-\lambda) u_e^t \end{aligned} \tag{5.14}$$

as a generalised flow cover inequality for the sets $CS^{BI}(u)$ and $CS^{UN}(u)$, where $\bar{c} = \max(c^t)_{t\in T}$.

We conclude this section with an example.

Example 5.12 *Consider a bounded cut set with UNdirected capacity constraints:*

$$\begin{aligned} CS^{UN}(u) = \text{conv}\{x \in \mathbb{Z}^2, f \in \mathbb{R}^8 \mid\ & f_1^1 + f_2^1 - f_3^1 - f_4^1 = 3, \\ & f_1^2 + f_2^2 - f_3^2 - f_4^2 = 1, \\ & 0 \leq f_1^1 + f_3^1 + f_1^2 + f_3^2 \leq 2x_1^1 + 3x_1^2, \\ & 0 \leq f_2^1 + f_4^1 + f_2^2 + f_4^2 \leq 2x_2^1 + 3x_2^2\} \end{aligned}$$

First note that we can calculate the flow cut inequalities (4.33) *that are obtained with a* MIR *procedure without complementing as described in Section 4.3.2. Many of those inequalities are strong for* $CS^{UN}(u)$ *but we will concentrate on the derivation of flow cover inequalities.*

Since $CS^{UN}(u)$ *is a cut set with UNdirected capacity constraints, we only choose base inequalities with* $E_1 \cap E_2 = \emptyset$ *and set* $\bar{c} = \max(c^t)_{t\in T} = 3$*. First let* $Q = \{1\}$*,* $E_1 = \{1\}$ *and* $E_2 = \emptyset$*. The corresponding base inequality of type* (4.31) *is*

$$f_2^1 + 2x_1^1 + 3x_1^2 \geq 3.$$

Setting $T_1 = \{1, 2\}$ *and* $(C_1, C_2) = (\{1\}, \emptyset)$ *gives a flow cover with* $\lambda = 2$*. The corresponding flow cover inequality is given by*

$$f_2^1 + x_1^2 \geq 1$$

defining a facet of $CS^{UN}(u)$*.*

Now take $Q = K^+ = \{1, 2\}$*. Let* $E_1 = \{1\}$ *and* $E_2 = \{2\}$*. The base inequality of type* (4.31) *is*

$$(f_2^1 + f_2^2) + \bar{f}_4^{1,2} + 2x_1^1 + 3x_1^2 - 2x_2^1 - 3x_2^2 \geq 4,$$

where $\bar{f}_4^{1,2} = 2x_2^1 + 3x_2^2 - f_4^1 - f_4^2$*. Set* $C_1 = \{1\}$*,* $C_2 = \emptyset$ *and* $T_1 = \{1, 2\}$*. It follows that* $\lambda = 1$ *and the flow cover inequality is*

$$f_2^1 + f_2^2 + x_2^1 + x_2^2 - f_4^1 - f_4^2 + x_1^1 + 2x_1^2 \geq 3$$

defining a high dimensional face of $CS^{UN}(u)$ *(dimension* ≥ 5*). Now set* $E_1 = \{1, 2\}$ *and* $E_2 = \emptyset$*. The base inequality is*

$$2x_1^1 + 3x_1^2 + 2x_2^1 + 3x_2^2 \geq 4.$$

Choosing $C_1 = \{1, 2\}$ *and* $T_1 = T_2 = \{2\}$ *means complementing* x_1^2, x_2^2*. The corresponding flow cover inequality with* $\lambda = 2$ *is*

$$x_1^1 + x_1^2 + x_2^1 + x_2^2 \geq 2.$$

defining a facet of $CS^{UN}(u)$*. The same cut inequality is obtained by calculating the* $\frac{1}{2}$-MIR *inequality for* $2x_1^1 + 3x_1^2 + 2x_2^1 + 3x_2^2 \geq 4$ *(without complementing).*

5.3 Summary

In this chapter we introduced the terms flow cover and flow pack. We started with a literature review in Section 5.1 and showed how well-known flow cover- and flow pack inequalities can be derived with a *MIR* procedure that extend the ones given in Chapter 4 by additionally complementing variables in an appropriate chosen flow cover. It then turned out that in the single-facility case this extended procedure is not useful for the sets considered in this thesis, which are based on arc independent capacity constraints. This is not true if more than one facility is considered.

In Section 5.2 it was shown that the extended *MIR* procedure to obtain flow cover inequalities in the general multi-facility case leads to a new class of strong valid inequalities. *MIR*-flow cover inequalities may be used to strengthen the initial formulation in Branch & Cut algorithms in addition to the pure flow cut inequalities of Chapter 4 obtained with the restricted procedure.

Chapter 6

Extensions and outlook

6.1 Introduction

In the last two chapters we made use of a general *MIR* procedure that has already been developed in Chapter 3. By aggregating inequalities of the initial formulation with respect to a cut of the network and applying *MIR*-procedure to the resulting valid inequalities it was shown how to detect strong valid and even facet-defining inequalities. In this chapter we concentrate on relaxations of network design polyhedra different from cut sets and apply the same *MIR* procedure. All the stated inequalities are well-known and most of them are facet-defining under certain conditions. We only give a review without providing any proofs elaborating the way to apply the *MIR* procedure. As an outlook on future research we pose some open questions and sketch some interesting ideas.

Literature Review and outline of this chapter Single arc (or edge) sets arise from the capacity constraint of a single arc (or edge). These sets and the corresponding arc (or edge) residual capacity inequalities for different models have been investigated by Magnanti et al. [1993, 1995], Rajan & Atamtürk [2002*b*] and Hoesel et al. [2000, 2004]. We will consider them in Section 6.2.

As a generalisation to cut inequalities we consider so-called multi cut inequalities in Section 6.3. Multi cut inequalities for partitions of size three have been studied by Magnanti et al. [1993] (UNdirected), Bienstock & Günlük [1996] (BIdirected) and Bienstock et al. [1995] (DIrected) for up to two facilities. All these articles are based on the study of complete three-node networks.

Section 6.4 reviews the results of Pochet & Wolsey [1992, 1995] on (unbounded) integer knapsack sets with divisible coefficients. The corresponding knapsack partition inequalities can be seen as multiple step *MIR*-inequalities.

Mixing and sequential pairing of *MIR* inequalities was introduced by Günlük & Pochet [2001]. Extensions are from Guan et al. [2004]. Günlük [1999] shows the usefulness of this approach for network design problems. The main results are given in Section 6.5.

We conclude with a note on sparse networks and cut set inequalities in Section 6.6.

6.2 Arc residual capacity inequalities

For simplicity we assume that demands are disaggregated (see Chapter 2), hence every commodity is given as a single point-to-point demand. Given the set of demand arcs D and a commodity $k =$

$(u, v) \in D = K$, define $d^k := d^k_u \in \mathbb{Z}_+\backslash\{0\}$. Hence d^k is the traffic that has to be routed from u to v for $k = (u, v)$. If Q is a subset of the commodities K set $d^Q := \sum_{k\in K} d^k$.

Consider a DIrected network design problem. Given a single arc a of the supply graph we can upper bound the flow on a by d^k for every commodity $k \in K$ since we can always delete flow around cycles:

$$f^k_a \leq d^k \qquad \forall k \in K, a \in A.$$

Assume that these constraints are added to the initial formulation which does not affect the optimal solution. Given $Q \subseteq K$ we additionally incorporate the (relaxed) capacity constraint of the arc a:

$$f^Q_a \leq \sum_{t\in T} c^t x^t_a$$

Setting $\widetilde{f}^Q_a := d^Q - f^Q_a \geq 0$ the following valid base inequality can be formulated:

$$\widetilde{f}^Q_a + \sum_{t\in T} c^t x^t_a \geq d^Q.$$

Choosing $s \in T$, applying *MIR* and restating in terms of the original flow variables gives so-called *arc residual capacity inequalities*:

$$-f^Q_a + \sum_{t\in T} \mathcal{G}_{d^Q,c^s}(c^t)x^t_a \geq r^Q_s \eta^Q_s - d^Q.$$

with $r^Q_s := r(d^Q, c^s)$ and $\eta^Q_s = \lceil \frac{d^Q}{c^s} \rceil$. Hence arc residual capacity inequalities can be obtained by considering a single arc of the network and a procedure that consists of **Substituting**, **Scaling** and ***MIR***.

The same procedure can now be applied to BIdirected and UNdirected problems. For BIdirected problems we consider the following two base inequalities defined for a single edge $e = ij$ of the network and a subset Q of the commodities

$$\widetilde{f}^Q_{ij} + \sum_{t\in T} c^t x^t_e \geq d^Q \quad \text{and} \quad \widetilde{f}^Q_{ji} + \sum_{t\in T} c^t x^t_e \geq d^Q$$

with $\widetilde{f}^Q_{ij} = d^Q - f^Q_{ij} \geq 0$ and $\widetilde{f}^Q_{ji} = d^Q - f^Q_{ij} \geq 0$ and for UNdirected problems

$$\widetilde{f}^Q_{ij} + \widetilde{f}^Q_{ji} + \sum_{t\in T} c^t x^t_e \geq d^Q$$

with $\widetilde{f}^Q_{ij} + \widetilde{f}^Q_{ji} = d^Q - f^Q_{ij} - f^Q_{ji}$. The resulting edge residual capacity inequalities are

$$\begin{aligned}
\text{BIdirected}: \quad & -f^Q_{ij} & + \sum_{t\in T} \mathcal{G}_{d^Q,c^s}(c^t)x^t_e \geq r^Q_s \eta^Q_s - d^Q,\\
& -f^Q_{ji} & + \sum_{t\in T} \mathcal{G}_{d^Q,c^s}(c^t)x^t_e \geq r^Q_s \eta^Q_s - d^Q\\
\text{UNdirected}: \quad & -f^Q_{ij} - f^Q_{ji} & + \sum_{t\in T} \mathcal{G}_{d^Q,c^s}(c^t)x^t_e \geq r^Q_s \eta^Q_s - d^Q
\end{aligned}$$

For all arc (or edge) residual capacity inequalities the coefficients of design variables can be rounded down to $\min\left(\mathcal{G}_{d^Q,c^s}(c^t), r_s^Q \eta_s^Q\right)$.

Edge residual capacity inequalities were introduced by Magnanti et al. [1993, 1995] for UNdirected problems with one and two facilities. We have generalised this class of inequalities to the multi-facility case.

Outlook By introducing the variables $y_a^k := \frac{f_a^k}{d^k}$, upper bound and capacity constraints reduce to

$$\sum_{k \in Q} d^k y_a^k \leq \sum_{t \in T} c^t x_a^t \quad \text{and} \quad y_a^k \leq 1$$

The (splittable flow) single arc set (polyhedron) for a is defined as the convex hull of

$$\{\, (y, x) \in [0,1]^Q \times \mathbb{Z}_+^T : \sum_{k \in Q} d^k y_a^k \leq \sum_{t \in T} c^t x_a^t \,\}$$

For non-bifurcated routing (unsplittable flow) y has to be restricted to $\{0,1\}^Q$.

For a study of such polyhedra see Magnanti et al. [1995], Rajan & Atamtürk [2002*b*] and Hoesel et al. [2000, 2004]. It is obvious that arc (or edge) residual capacity inequalities are valid for the corresponding arc (or edge) set polyhedra. They are facet-defining under certain conditions. This is proven at least for the one and two facility case (with divisible capacities).

Magnanti et al. [1995] show that edge residual capacity inequalities can even be facet-defining for NDP^{UN} (with two divisible facilities). Their result might be decomposable. As in Theorem 4.4 one might find conditions under which facet-defining inequalities for arc (or edge) set polyhedra are facet-defining for the corresponding network design polyhedra. Hoesel et al. [2000, 2004] did so for BIdirected and UNdirected problems but with non-bifurcated routing.

Applying *MIR* as above is equivalent to considering arc (or) edge residual capacity inequalities in the single-facility case and then lift them to the multi-facility case by using the valid subadditive *MIR* function $\mathcal{G}$ (see Chapter 3). It is not clear if this lifting is exact.

Rajan & Atamtürk [2002*b*] state a linear-time separation procedure for arc residual capacity inequalities in the single-facility case. This result might be extendable to the multi facility case.

6.3 Multi cut inequalities

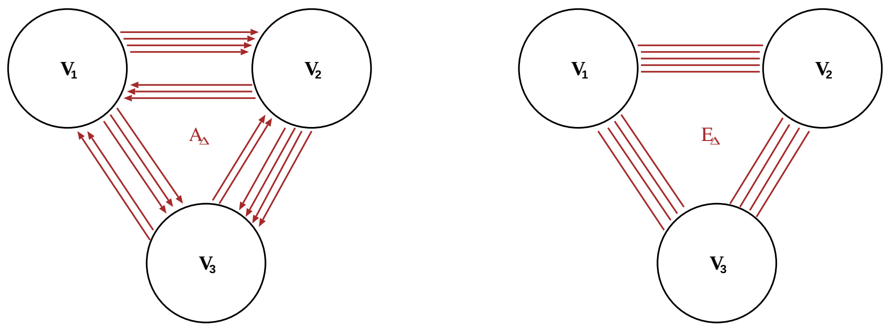
Let $\triangle := \{V_1, ..., V_m\}$ be a (disjoint) partition of the nodes V, with $m \in \mathbb{Z}_+ \backslash \{0\}$ and $m \geq 3$. A multi cut is the set of all arcs (or edges) with endnodes not both in one the sets $V_1, ..., V_m$. For directed supply graphs we define the multi cut

$$A_\triangle := \bigcup_{i=1}^{m} A_{V_i}$$

and similar for undirected graphs let

$$E_\triangle := \bigcup_{i=1}^{m} E_{V_i}$$

be the multi cut corresponding to $\triangle$.



(i) directed multi cut $A_\triangle$ (ii) undirected multi cut $E_\triangle$

Figure 6.1: Multi cut $\triangle = \{V_1, V_2, V_3\}$

It will be explained how to derive a multi cut inequality for DIrected problems. At the end we will state the corresponding inequalities for BIdirected and UNdirected problems. Consider an element k of the set $2^\triangle$ of all subsets of $\triangle$ that is not empty and that is not $\triangle$. Now let $S_k \subset V$ be the node set that is the union of all elements in k. There are $2^m - 2$ such node sets. Given the cut A_{S_k}, recall the base inequality

$$\sum_{t\in T} c^t x^t(A^+_{S_k}) \geq d^+_k$$

that was used in Section 4.2.1 and Section 4.3.1 to derive cut inequalities for CS^{DI}, where $d^+_k = d^{K+}_{S_k}$ denotes the total demand from S_k to $V\backslash S_k$. Now for all k in $2^\triangle$ (not empty and not $\triangle$) we simply sum up all these base inequalities resulting in

$$2^{m-2}\sum_{t\in T} c^t x^t(A_\triangle) \geq \sum_{k\in 2^\triangle} d^+_k.$$

What we have done is enumerating all cuts that correspond to the partition $\triangle$ and summing up all the corresponding (base) cut inequalities. This way we count every link design variable for $\triangle$ and every single demand exactly 2^{m-2} times. By setting

$$d_\triangle := \frac{\sum_{k\in 2^\triangle} d^+_k}{2^{m-2}} \in \mathbb{Z}_+$$

we get

$$\sum_{t\in T} c^t x^t(A_\triangle) \geq d_\triangle$$

that we will use as a multi cut base inequality ($m \geq 3$). Similar to the procedure in Section 4.3.1 we can now use the *MIR* function $\mathcal{G}_{d_\triangle, c^s}$ for every $s \in T$ to obtain a multi cut inequality

$$\sum_{t\in T} \mathcal{G}_{d_\triangle, c^s}(c^t) x^t(A_\triangle) \geq r^\triangle_s \eta^\triangle_s \tag{6.1}$$

where $r^\triangle_s = r(d_\triangle, c_s)$ and $\eta^\triangle_s = \lceil \frac{d_\triangle}{c_s} \rceil$.

Applying the same procedure that consisted of the steps **Aggregating**, **Scaling** and ***MIR*** to the base cut inequalities

$$\sum_{t\in T} c^t x^t(E_{S_k}) \geq \max(d_k^+, |d_k^-|)$$

and

$$\sum_{t\in T} c^t x^t(E_{S_k}) \geq d_k^+ + |d_k^-| \tag{6.2}$$

for BIdirected and UNdirected problems respectively, results in the multi cut inequality

$$\sum_{t\in T} \mathcal{G}_{d_\triangle, c^s}(c^t) x^t(E_\triangle) \geq r_s^\triangle \eta_s^\triangle \tag{6.3}$$

$$\text{with} \quad d_\triangle := \left\lceil \frac{\sum_{k\in 2^\triangle} \max(d_k^+, |d_k^-|)}{2^{m-1}} \right\rceil \in \mathbb{Z}_+ \quad \text{for BIdirected problems and}$$

$$d_\triangle := \frac{\sum_{k\in 2^\triangle} (d_k^+ + |d_k^-|)}{2^{m-1}} \in \mathbb{Z}_+ \quad \text{for UNdirected problems.}$$

The multi cut inequalities (6.1) and (6.3) might even be strengthened as in Section 4.3.1 by rounding down coefficients to the value of the right hand side and considering $\min\left(\mathcal{G}_{d_\triangle, c^s}(c^t), r_s^\triangle \eta_s^\triangle\right)$.

Note that for BIdirected problems $\frac{\sum_{k\in 2^\triangle} \max(d_k^+, |d_k^-|)}{2^{m-1}}$ is not necessarily integer. We can round it up before the *MIR* step. The value $|d_k^-| = |d_{S_k}^{K-}|$ denotes the total demand from $V\backslash S_k$ to S_k. Inequality (6.2) is obtained by applying a transformation to the cut variables as described in Section 4.1 to ensure that there are only positive commodities with respect to the cut E_{S_k}.

It is possible to strengthen the value $d_\triangle$ for all three types of capacity usage. One has to consider so-called *metric inequalities* (see for instance Günlük [1999]) and to incorporate shortest paths between demand endnodes with respect to weights that are defined by the incidence vector of the multi cut (see also Section 6.6). This will not be done here.

Outlook Bienstock & Günlük [1996] investigate multi cut inequalities for $m = 3$ of type (6.3) and show that they can be facet-defining for the network design polyhedron NDP^{BI}. An important condition in this context is that for all node sets S_i of the partition it holds that both subgraphs $G[S_i]$ and $G[V\backslash S_i]$ are connected. With this result in mind it could be interesting to study *multi cut set polyhedra* as a generalisation of cut set polyhedra and to develop a generalisation of Theorem 4.4.

Another interesting idea in the context of multi cuts is to apply the above aggregation and *MIR* procedure to flow cut inequalities resulting in *multi flow cut inequalities*. This has been done first by Rajan & Atamtürk [2002*a*, 2004]. The question that arises is when those inequalities are facet-defining for multi cut set polyhedra and network design polyhedra.

6.4 Divisible coefficients and (multi) cut inequalities

Suppose that the set of technologies T is given by divisible base capacities and consider a network cut A_S with $S \subset V$ and the corresponding base cut inequality

$$\sum_{t\in T} c^t x^t(A_S^+) \geq d_S^{K+}.$$

This base inequality defines a so-called *integer knapsack set* of the form

$$X = \{\, x \in \mathbb{Z}^n_+ :\ \sum_{j=1}^{n} c^j x^j \geq d,\ \ c^j, d \in \mathbb{Z}_+ \backslash \{0\}\, \}$$

where $\frac{c^{j+1}}{c^j} \in \mathbb{Z}_+ \backslash \{0\}$ and $c^1 = 1$ w. l. o. g..

Pochet & Wolsey [1992, 1995] investigate such sets and are able to give a complete description of $\mathrm{conv}(X)$ by considering so-called *knapsack partition inequalities*. We will state a multi step *MIR* procedure that produces knapsack partition inequalities (without a proof).

Let us briefly review the results and the notation of Pochet & Wolsey [1992, 1995]: Consider partitions of $\{1, ..., n\}$ into blocks

$$\{i_1, ..., j_1\}\{i_2, ..., j_2\}, ..., \{i_p, ..., j_p\} \quad \text{with} \quad i_1 = 1, j_p = n, i_t = j_{t-1} + 1 \quad \text{for} \quad t = 2, ..., p.$$

Define β_t, κ_t for every block:

$$\beta_p := d, \quad \kappa_t := \lceil \frac{\beta_t}{c^{i_t}} \rceil, \quad \beta_{t-1} := \beta_t - (\kappa_t - 1)c^{i_t}.$$

Note that by Lemma 3.11 $\beta_{t-1} = r(\beta_t, c^{i_t})$. A knapsack partition inequality is now given by

$$\sum_{t=1}^{p} (\prod_{s=1}^{t-1} \kappa_s) \sum_{j=i_t}^{j_t} \min(\frac{c^j}{c^{i_t}}, \kappa_t) x_j \geq \prod_{t=1}^{p} \kappa_t. \tag{6.4}$$

$\mathrm{conv}(X)$ is completely described by the non-negativity constraints and (6.4) with respect to all possible partitions of the index set $\{1, ..., n\}$. Let us first concentrate on partitions into two blocks $\{i_1, ..., j_1\}$ and $\{i_2, ..., j_2\}$. It follows that $\eta := \kappa_2 = \lceil \frac{d}{c^{i_2}} \rceil$ and $r := \kappa_1 = r(d, c^{i_2})$. Hence (6.4) reduces to

$$\sum_{j=i_1}^{j_2} \min(c^j, r) x_j + \sum_{j=i_2}^{j_2} \min(r\frac{c^j}{c^{i_2}}, r\eta) x_j \geq r\eta.$$

But

$$\min(c^j, r) = r - \left(r - c^j\right)^+ = \mathcal{G}_{d, c^{i_2}}(c^j) \leq r\eta$$

since $c^j \leq c^{i_2}$ for $j < i_2$. Similarly,

$$\min(r\frac{c^j}{c^{i_2}}, r\eta) = \min\left(\mathcal{G}_{d, c^{i_2}}(c^j), r\eta\right)$$

for $j \geq i_2$. The latter follows from the fact that the capacities are divisible. Hence (6.4) can be obtained by *MIR* with scaling factor $\frac{1}{c^{i_2}}$-*MIR* plus rounding down the coefficients to the value of the right hand side.

It follows that if the integer knapsack set is given by a base cut inequality or by a base multi cut inequality, then knapsack partition inequalities for partitions that consists of two blocks reduce to the strengthened (multi) cut inequalities of Section 4.3 and 6.3.

If now the partition of the index set consists of p blocks, then the corresponding knapsack partition inequality is obtained by a $p-1$-step *MIR* procedure. We state an example taken from Magnanti & Mirchandani [1993] which we already considered in Section 4.3.2.

Example 6.1 (continued from Example 4.36) *Again consider a network design polyhedron with UNdirected capacity constraints, three facilities and one commodity, where* $c^1 = 1$, $c^2 = C \in \mathbb{Z}_+, C > 1$ *and* $c^3 = \lambda C \in \mathbb{Z}_+, \lambda > 1$. *Given a cut* E_S *we could formulate two strengthened cut inequalities with the* MIR *procedure of Section 4.3.2:*

$$x^1(E_S) + r_1 x^2(E_S) + \min(r_1\lambda, r_1\lceil \tfrac{d}{C}\rceil) x^3(E_S) \geq r_1\lceil \tfrac{d}{C}\rceil. \tag{6.5}$$

and

$$x^1(E_S) + \min(C, r_2) x^2(E_S) + r_2 x^3(E_S) \geq r_2\lceil \tfrac{d}{\lambda C}\rceil, \tag{6.6}$$

with $r_1 = r(d, C)$ *and* $r_2 = r(d, \lambda C)$ *These are knapsack partition inequalities. They correspond to partitions of the index set into two blocks where* $c^{i_2} = C$ *or* $c^{i_2} = \lambda C$.

There is a third possible partition (giving a new knapsack partition inequality) that consists of three blocks with $c^{i_2} = C$ *and* $c^{i_3} = \lambda C$. *We apply a two-step* MIR *procedure to the base cut inequality. First we divide by* $c^{i_2} = C$ *and apply* MIR *as it has been done for* (6.5) *resulting in*

$$x^1(E_S) + r_1 x^2(E_S) + r_1\lambda x^3(E_S) \geq r_1\lceil \tfrac{d}{C}\rceil.$$

This is inequality (6.5) *but without rounding down coefficients the value of the right hand side. Note that the coefficients of this inequality again are divisible. Now we divide by* $\mathcal{G}_{d,c^{i_2}}(c^{i_3}) = r_1\lambda$ *and apply* MIR. *This gives*

$$x^1(E_S) + r_1 x^2(E_S) + r_1 r(\lceil \tfrac{d}{C}\rceil, \lambda) x^3 \geq r_1 r(\lceil \tfrac{d}{C}\rceil, \lambda)\lceil \tfrac{d}{\lambda C}\rceil \tag{6.7}$$

Note that $\lceil \frac{r_1\lceil \frac{d}{C}\rceil}{r_1\lambda}\rceil = \lceil \frac{d}{\lambda C}\rceil$ *and* $r(r_1\lceil \frac{d}{C}\rceil, r_1\lambda) = r_1 r(\lceil \frac{d}{C}\rceil, \lambda)$. *The same inequality is obtained when evaluating* $\kappa_1, \kappa_2, \kappa_3$ *and calculating* (6.4).

The last example suggests that the partition $\{i_1, ..., j_1\}\{i_2, ..., j_2\}, ..., \{i_p, ..., j_p\}$ defines a sequence for the application of a $p-1$ *MIR* procedure to obtain the knapsack partition inequality (6.4):

We start with the base inequality $c^j x^j \geq d$. If $\bar{c}$ defines the vector of coefficients after the step $k-1$ of the procedure, then we divide the *MIR* inequality by $\bar{c}^{i_{k+1}}$ in step k and apply *MIR* again. The resulting *MIR* inequality serves as the base inequality for the next step. After step $p-1$ of the procedure all coefficients get rounded down to the value of the right hand side (if greater) resulting in the knapsack partition inequality (6.4).

Outlook It is obvious that the $p-1$ step *MIR* procedure above can also be applied to *mixed integer knapsack sets* defined by a single constraint of the form $f + c^j x^j \geq d$ with f continuous. It follows that the procedure might be useful when applied to simple flow cut inequalities similar to the way we applied it to cut inequalities. Moreover, the procedure can be applied even if the coefficients are not divisible. It is an open question if in these cases the corresponding (mixed) knapsack partition inequalities define facets of the (mixed) integer knapsack sets or even the network design polyhedra. It it also unknown under which conditions knapsack partition inequalities derived from base cut inequalities and defined for partitions of the index set greater than two define facets for cut sets.

6.5 Mixing *MIR* and mixing cut set inequalities

MIR as introduced in Chapter 3 is applied to a single base inequality. The idea of mixing is now to consider more than one base inequality at once. This is due to Günlük & Pochet [2001] and there are some new results given in Guan et al. [2004].

There are many special cases and extensions. Only a motivation will be given here. Let $X \subseteq \mathbb{R}^M_+ \times \mathbb{Z}^N_+$ be a mixed integer set. Consider the following two functions: $g^i : X \to \mathbb{R}_+$ and $h^i : X \to \mathbb{Z}_+$ with $i \in J := \{1, ..., n\}$. Let $c, d^i \in \mathbb{R}_+$ and $r^i := r(d^i, c)$. We have a collection of n valid inequalities for X:

$$g^i(f, x) + ch^i(f, x) \geq d^i, \qquad \forall i \in J, (f, x) \in X$$

Lets assume that $r^i \geq r^{i-1}$ and $r^0 := 0$

Theorem 6.2 (Günlük & Pochet [2001]) *If $g : X \to \mathbb{R}_+$ and $g(f, x) \geq g^i(f, x)$ for all $i \in \{1, ..., n\}$ and $(f, x) \in X$ then the following mixed* MIR *inequalities are valid for X:*

$$g(f, x) \geq \sum_{i=1}^{n} (r^i - r^{i-1})(\lceil \frac{d^i}{c} \rceil - h^i(f, x)), \tag{6.8}$$

$$g(f, x) \geq \sum_{i=1}^{n} (r^i - r^{i-1})(\lceil \frac{d^i}{c} \rceil - h^i(f, x)) + (c - r^n)(\lceil \frac{d^1}{c} \rceil - h^1(f, x) - 1). \tag{6.9}$$

Note that if $|J| = 1$ and $g = g^1$ then (6.8) reduces to the *MIR* inequality (3.10) and (6.9) reduces to the base inequality again. So the last theorem provides a generalisation of *MIR* to a collection of n base inequalities.

Günlük [1999] shows how to exploit this new result for network design problems. As a simple example assume to have a BIdirected problem with exactly two facilities and divisible capacities. So we can set $c^1 = 1$ and $c^2 = \lambda$. Now for every $i \in J$ consider a cut E_{S_i} of the network and the corresponding base cut inequality

$$x^1(E_{S_i}) + \lambda x^2(E_{S_i}) \geq \max(d^{K+}_{S_i}, |d^{K-}_{S_i}|).$$

By setting $g^i(f, x) := x^1(E_{S_i})$ and $h^i(f, x) := x^2(E_{S_i})$ this collection of base inequalities fulfils the conditions of Theorem 6.2. It remains to define a function g with $g \geq g^i$ for all $i \in J$. We can simply sum up all functions g^i. But this will make the left hand side large and might result in weak mixed *MIR* inequalities.

A wonderful idea of Günlük [1999] is to consider a three partition $\triangle := \{S_1, S_2, S_3\}$ of the network and all three cuts E_{S_1}, E_{S_2} and E_{S_3}. A canonic function g with $g \geq g^1, g^2, g^3$ is now given by $g(f, x) = x^1(E_\triangle)$. Moreover, the corresponding base multi cut inequality

$$x^1(E_\triangle) + \lambda x^2(E_\triangle) \geq d_\triangle$$

provides a fourth base inequality. Günlük [1999] proves that under certain conditions mixing two of the three cut inequalities corresponding to a three partition or mixing a (three) multi cut inequality with one of the three corresponding cut inequalities results in facet-defining mixed *MIR* inequalities (6.8) for the polyhedron NDP^{BI} with two divisible facilities.

Outlook Mixing *MIR* is somewhat restricted to a single coefficient c and hence can be applied easily to single-facility network design problems or those with two divisible facilities. But for the multi-facility case more general formulas than (6.8) and (6.9) are missing.

It has not been tried yet to mix flow cut inequalities that for instance arise when considering all cuts of a three partition.

6.6 A note on sparse networks

In this thesis we have mainly concentrated on cuts of the network and the corresponding cut set polyhedra to develop strong valid *MIR*-inequalities. By Theorem 4.4, an important condition for the strength of cut set inequalities is a certain connectivity of the two subgraphs $G[S]$ and $G[V\setminus S]$ defined by the cut. None of the considered inequalities in this thesis does exploit the structure of these subgraphs. This might be a drawback when optimising sparse networks as they are common in practice. In Chapter 7 we can still prove the usefulness of the investigated cut set inequalities for real-life networks but facing the fact that our separation heuristics are very fast, it might be worth to spend more time for incorporating the structure of $G[S]$ and $G[V\setminus S]$.

There is not much research on how to strengthen cut set inequalities for sparse graphs. Some ideas can be found in Ortega & Wolsey [2003]. Given a cut A_S and a single commodity $k = (u, v)$ in K^+, they are able to strengthen cut as well as simple flow cut inequalities for uncapacitated, fixed-charge network design problems by considering subsets of the dicut arcs A_S that are reachable from the demand endnodes u and v.

Bienstock et al. [1995] and Bienstock & Günlük [1996] explicitly calculate subsets S with the property that $G[S]$ and $G[V\setminus S]$ are strongly connected (directed graphs) or connected (undirected graphs). They do this in addition to fast (contraction) heuristics as those used in Chapter 7 and report good results for some of their considered networks.

6.7 Summary

It was shown that the *MIR* procedure introduced in Chapter 3 is not restricted to network cuts. There are various classes of strong valid inequalities for network design polyhedra that can be derived. Similar to cut sets and cut set inequalities all stated inequalities correspond to relaxations obtained by considering certain network structures. For multi cut inequalities, arc residual capacity inequalities and knapsack partition inequalities we could directly apply our *MIR* procedure, whereas for mixing *MIR* several base inequalities had to be considered at once generalising the procedure used in the last chapters.

It might be possible to strengthen the base inequalities for all classes of strong valid *MIR*-inequalities considered in this thesis if the underlying graphs are sparse.

Chapter 7

Separation, Implementation and Computational Results

7.1 Introduction

Due to time limits for this thesis, this chapter mainly addresses the separation and implementation of the cut set inequalities investigated in Chapter 4 but we will discuss the usefulness of those inequalities both for bounded and unbounded network design polyhedra.

Literature review and complexity of separation Mirchandani [1989] proves that finding a node set S that gives a violated cut inequality in the single-commodity case (single source (s), single sink (t)) is a *max flow* problem, for which polynomial time algorithms exist (Schrijver [2003]). Atamtürk [2002] shows that the generalised problem of finding a violated flow cut inequality is equivalent to a s-t *mincut* problem with negative weights on some arcs, which is $\mathcal{NP}$-hard, having the s-t *maxcut* problem as a special case (Garey & Johnson [1979]). Remember that cut inequalities form a subclass of general flow cut inequalities. The multi-commodity case is $\mathcal{NP}$-hard even for cut inequalities, again by reduction to the *maxcut* problem (Baharona [1994]).

There is no literature about the simultaneous determination of the sets S, Q, A_1^+ and A_2^- to find violated flow cut inequalities. All approaches are based on decomposing the separation procedure. The effectiveness of cut inequalities for network design problems within a Branch & Cut framework was investigated by Magnanti et al. [1995], Bienstock et al. [1995], Bienstock & Günlük [1996], Günlük [1999] and Atamtürk [2002]. All of them use heuristics for the separation of a node set S. Bienstock et al. [1995], Bienstock & Günlük [1996] and Atamtürk [2002] consider the separation of (simple) flow cut inequalities and implemented different heuristics for the determination of appropriate commodity subsets Q, given a fixed node set S. Atamtürk [2002] is the first to state an exact polynomial time algorithm for the separation of the arc sets A_1^+ and A_2^-, given a fixed cut of the network and a fixed commodity subset.

Outline of this chapter We will first recall the inequalities developed in Chapter 4 that have been used in the implementation. The separation problem will be defined in Section 7.2 and we will show how to decompose it, which motivates a separation algorithm consisting of heuristics for the determination of node sets and commodity subsets and an exact procedure that computes arc sets (edge sets) of the considered cut. When developing this algorithm we make use of the results of Chapter 4.

Section 7.3 addresses some more detailed implementational aspects. We will discuss some modifications of the initial separation algorithm and in Section 7.4 our approach will be tested against real world network design problems.

Inequalities We consider the following strong valid inequalities for network design polyhedra. In Chapter 4 we studied general flow cut inequalities of the form:

NDP^{DI}:

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t)x^t(A_1^+) + \sum_{t\in T}(c^t + \mathcal{G}_{d,c^s}(-c^t))x^t(A_2^-) \geq r_s^Q \eta_s^Q \tag{7.1}$$

NDP^{BI} and NDP^{UN}:

$$f^Q(\bar{E}_1^+) - f^Q(E_2^-) + \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t)x^t(E_1) + \sum_{t\in T}(c^t + \mathcal{G}_{d,c^s}(-c^t))x^t(E_2) \geq r_s^Q \eta_s^Q, \tag{7.2}$$

with $r_s^Q = r(d_S^Q, c^s)$ and $\eta_s^Q = \lceil \frac{d_S^Q}{c^s} \rceil$. A general flow cut inequality is defined by a node set $S \subset V$, a subset Q of the commodities K, arc sets $A_1^+ \subseteq A_S^+$ and $A_1^+ \subseteq A_S^-$ (edge sets $E_1, E_2 \subseteq E_S$) and a facility $s \in T$.

We restrict our attention to $Q \subseteq K^+$ and $Q \subseteq K^-$ because in Chapter 4 we proved facet theorems for these cases. In Section 4.3 it was shown that flow cut inequalities can be strengthened if $A_2^- = \emptyset$ ($E_2 = \emptyset$) by rounding down coefficients to the value of the right hand side, leading to (strengthened) simple flow cut inequalities:

NDP^{DI}:

$$f^Q(\bar{A}_1^+) + \sum_{t\in T} \min(r_s^Q \eta_s^Q, \mathcal{G}_{d,c^s}(c^t))x^t(A_1^+) \geq r_s^Q \eta_s^Q \tag{7.3}$$

NDP^{BI} and NDP^{UN}:

$$f^Q(\bar{E}_1^+) + \sum_{t\in T} \min(r_s^Q \eta_s^Q, \mathcal{G}_{d,c^s}(c^t))x^t(E_1) \geq r_s^Q \eta_s^Q \tag{7.4}$$

Eventually the new cut set inequality

$$cx(E_1) + \left(c - r^{K^+}\right)x(\bar{E}_1) + f^{K^+}(E_1^-) - f^{K^+}(E_1^+) \geq c - r^{K^+} \tag{7.5}$$

with $r^{K^+} = r(d_S^{K^+}, c)$ will be considered. It was introduced in Section 4.2.2 for BIdirected and UNdirected problems in the single facility case and we stated a facet proof (By switching to $V\backslash S$ if necessary, we assume that $d_S^{K^+} \geq |d_S^{K^-}|$). A generalisation of these inequalities to the multi-facility case (by *MIR*) is not known but it might still be interesting to use them for test instances with $|T| = 1$. For this cut set inequality we need to determine $S \subset V$ and $E_1 \subseteq E_S$.

7.2 Separation

Preliminaries The *separation problem* has already been sketched in the literature review. For completeness we will now state some definitions that will be used in this section. Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $\bar{p}$ a point in $\mathbb{R}^n$. The **separation problem** is now to decide whether $\bar{p} \in P$, and if not, to find a hyperplane that separates $\bar{p}$ from P, more precisely to find a vector $(a, d) \in \mathbb{R}^n \times \mathbb{R}$ such that the inequality $a^T x \geq d$ is valid for P but violated by the point $\bar{p}$. In general, given a valid inequality $a^T x \geq d$ for P and a point $\bar{p} \in \mathbb{R}^n$, we call $d - a^T \bar{p}$ the corresponding **violation**. Hence the point $\bar{p}$ is not in P if the violation is positive.

The euclidian **distance** from $\bar{p}$ to the hyperplane defined by $a^T x = d$ is given by

$$\frac{|d - a^T \bar{p}|}{\|d\|},$$

where $\|\cdot\|$ denotes the euclidian norm in $\mathbb{R}^n$.

Let $(a_1^T, d_1) \in \mathbb{R}^n \times \mathbb{R}$ and $(a_2^T, d_2) \in \mathbb{R}^n \times \mathbb{R}$ define two hyperplanes in R^n. Then its **orthogonality** $o(a_1, a_2)$ is given by

$$0 \leq o(a_1, a_2) := \frac{|a_1^T a_2|}{\|a_1\| \|a_2\|} \leq 1.$$

The hyperplanes are parallel if and only if its orthogonality equals zero and they are orthogonal if and only if its orthogonality is one. $1 - o(a_1, a_2)$ will be called the **parallelism** of the two hyperplanes.

In fact we are not faced with the general separation problem above but with the problem of finding a violated inequality from a class of valid inequalities for P. Let I be a finite index set and let $C = \{ (a_i, d_i) \in \mathbb{R}^n \times \mathbb{R} : i \in I \}$ define a class of valid inequalities for P:

$$a_i^T x \geq d_i \qquad \forall x \in P, i \in I.$$

Given a point $\bar{p}$, the separation problem for C is to find $i \in I$ such that $a_i^T \bar{p} < d_i$ or to decide that such an inequality in C does not exist. Note that the latter does not imply $\bar{p} \in P$. The separation problem for C is obviously equivalent to the problem of finding a **most violated** inequality in C, that is to determine $j \in I$ that maximises the violation with respect to $\bar{p}$:

$$(d_j - a_j^T \bar{p}) \;\geq\; (d_i - a_i^T \bar{p}) \quad \text{for all} \quad i \in I.$$

A most violated inequality always exists but it is not necessarily unique. Moreover, a most violated inequality is not violated by $\bar{p}$ if and only if there is no violated inequality in C.

Decomposition Given a network design polyhedron and a point $\bar{p} = (\bar{f}, \bar{x})$, the separation problem for (simple) flow cut inequalities or cut set inequalities of type (7.5) reduces to the problem of simultaneously determining a node set S, a commodity subset Q and arc- or edge sets of the cut $\delta(S)$ that give a most violated inequality. Due to the mentioned result of (Baharona [1994]) this problem is $\mathcal{NP}$-hard for all three classes of inequalities. Note that by setting $E_1 = E_S$, (7.2) can also be seen as a superclass of cut inequalities as shown in the proof of Theorem 4.29.

We will use the following decomposition approach as a separation heuristic for the mentioned inequalities:

- Heuristically compute a promising cut of the network (node set S).
- Given a node set S, heuristically compute a promising commodity subset Q
- Given a node set S and a commodity subset Q, compute a most violated (simple) flow cut inequality or a most violated cut set inequality (7.2).

Note that if the network cut is fixed, the complexity of separating flow cut inequalities is an open question even for the single-facility case. For some special cases see Atamtürk [2002]. In general we do not know an efficient way to choose commodity subsets that give most violated inequalities. But as we will see, it is possible to exactly separate the considered inequalities if both the cut and the commodity subset are fixed. There are many possible answers to the question, what is *promising*. Some of them will be considered in the following.

Although not tested, the author believes that this decomposition approach is even useful for the flow cover inequalities considered in Chapter 5. A cover (C^+, C^-) with $C^+ \subseteq A_1^+$ and $C^- \subseteq A_2^-$ has to be found in addition. One could extend the approach above by a fourth step, that consists of finding an appropriate cover.

Finding a node set S The most simple approach for finding cuts of the network that give violated flow cut inequalities is that of enumerating all node sets with a small number of nodes in S in each iteration of the cutting plane phase. This approach has been used by Magnanti et al. [1995] ($|S| \leq 5$) and Atamtürk [2002] ($|S| = 1$). It might be useful for small instances and at the beginning of the Branch & Cut algorithm. But once all violated flow cut inequalities corresponding to small sized node sets are added to the initial formulation, one has to use more general heuristics.

A more promising idea of Bienstock et al. [1995] and Bienstock & Günlük [1996] is to consider subsets S only if $G[S]$ and $G[V \setminus S]$ are connected (undirected graphs) or strongly connected (directed supply graphs). Recall from Section 4.1 that a cut set inequality is facet defining for a network design polyhedron if it defines a facet for the corresponding cut set and if $G[S]$ and $G[V \setminus S]$ are (strongly) connected. Bienstock et al. [1995] and Bienstock & Günlük [1996] call node sets S with that property *strong* (or *critical*) and enumerate all strong node sets at the beginning of the optimisation procedure (for networks with $|V| \leq 27$ and $|A| \leq 102$). Although there are potentially $2^{|V|}$ node sets to be considered, real-life networks are usually fairly sparse such that the number of strong node sets is limited, so it might be attractive to enumerate at least some of them.

A very fast and general separation heuristic has been proposed and successfully used by Bienstock et al. [1995] and Günlük [1999]. Since we basically use their approach in our implementation we explain it here in more detail. It is in fact the only one that exploits the values of the current primal solution $\bar{p} = (\bar{f}, \bar{x})$. The idea is that if on arcs of the cut the installed capacity is large compared to the current flow, then it its unlikely that a flow cut inequality is violated. Thus we concentrate on cuts that have few arcs with large slack of the corresponding capacity constraint. Define the arc weights:

$$\text{DIrected:} \quad w_a := \sum_{t \in T} c^t \bar{x}_a^t - \bar{f}_a^K \qquad a \in A$$

$$\text{BIdirected:} \quad w_e := \sum_{t \in T} c^t \bar{x}_e^t - \max(\bar{f}_{ij}^K, \bar{f}_{ji}^K) \qquad e = ij \in E$$

$$\text{UNdirected:} \quad w_e := \sum_{t \in T} c^t \bar{x}_e^t - (\bar{f}_{ij}^K + \bar{f}_{ji}^K) \qquad e = ij \in E.$$

In Algorithm 7.1 a **contraction** procedure is applied to the network using the weights w_a (w_e). The graph shrinks until it has exactly *PSize* nodes. We do that by contracting the endnodes of arcs with large slack. See Grötschel et al. [1988] for a thorough description of this operation.

In each step of the algorithm the set $\triangle$ contains a partition $V_1, \ldots, V_p$ of the nodes V with *PSize* $\leq p \leq |V|$. Assume *PSize* ≥ 2 and let V_u be the unique node set in $\triangle$ containing the node u. Algorithm 7.1 terminates with all cuts corresponding to the shrunken graph.

Input : point $\bar{p} = (\bar{f}, \bar{x})$, *PSize* ≥ 2, $G = (V, A)$ ($G = (V, E)$) connected with $|V| \geq$ *PSize*
Output : a list of $2^{PSize-1} - 1$ subsets of V, all corresponding to different cuts of the network

1: Calculate the slack weights w_a (or w_e) for all arcs (for all edges).
2: Prepare a list of all arcs (or edges) in decreasing order of w_a (or w_e).
3: Initialise $\triangle := \{ V_i : i \in V \}$ where $V_i := \{i\}$ for all $i \in V$
4: **while** $|\triangle| >$ *PSize* **do**
5:   Pop $a = (u, v)$ (or $e = uv$) from the top of the list of arcs (or edges).
6:   **if** $V_u \neq V_v$ **then**
7:     Contract the endnodes of arc a (or edge e): Let $\bar{V} = V_u \cup V_v$. Set $\triangle \leftarrow \triangle \cup \{\bar{V}\} \backslash \{V_u, V_v\}$.
8:   **end if**
9: **end while**
10: Considering the contracted graph defined by $\triangle$, enumerate all cuts and return the corresponding node sets as a list.

Algorithm 7.1: NODESETSBYCONTRACTION($\bar{p}$, *PSize*, G)

Note that step 2 is not unique. Arcs with the same slack value can be sorted arbitrarily (or randomly). Algorithm 7.1 works correctly since the contracted graph defined by $\triangle$ remains connected during the algorithm. Calculating the weights runs in $O(|A||T|)$ and sorting them in $O(|A| \log|A|)$. The computation time of the shrinking procedure is bounded by the number of arcs (assuming that contracting two nodes of a graph can be done in constant time). Enumerating the cuts of the final graph is exponential in *PSize*. But assuming *PSize* to be small and constant over all instances, the running time for Algorithm 7.1 is in $O(|A|(|T| + \log|A|)$ (resp. $O(|E|(|T| + \log|E|))$).

There are several possible extensions to Algorithm 7.1. Instead of using slack weights w_a, Günlük [1999] additionally considers the value of the dual variable π_a corresponding to the capacity constraint and the current fractional solution and uses weights $w_a^{'} := w_a - |\pi_a|$.

Another idea is that of "kicking" (Bienstock et al. [1995]). Given a node set S from the list returned by Algorithm 7.1, one can also check $S \backslash \{i\}$ or $S \cup \{i\}$ for violated flow cut inequalities, where $i \in V \backslash S$.

Input : node set $S \subset V$, net demands d_i^k for all $i \in V, k \in K$
Output : DIrected, BIdirected: a list L of positive and negative commodity subsets
UNdirected: a list L of positive commodity subsets

1: Calculate $d_S^k = \sum_{i \in S} d_i^k$ for all $k \in K$
2: **return** $L := \{K^+\} \cup \{K^-\} \bigcup_{k \in K^+ \cup K^-} \{k\}$

Algorithm 7.2: COMMODITYSUBSETS(S, d)

Finding a commodity subset Q In general no polynomial-time algorithm is known to find a proper commodity subset even if the arc sets are fixed.

All mentioned authors use very basic heuristics similar to the one that we use for our implementation. Given a fixed node set S, we only consider commodity subsets Q with an aggregated demand of $d_S^Q \neq 0$. We concentrate on singleton commodities and the sets K^+ and K^- as defined in Section 4.1. Those commodity subsets are put into a list by Algorithm 7.2 which runs in $O(|K||V|)$. Remember that for UNdirected problems we assume $K^- = \emptyset$.

Arc sets A_1^+, A_2^- or edge sets E_1, E_2 Atamtürk [2002] states a procedure that, given a fixed cut and a fixed commodity subset, exactly separates flow cut inequalities. The procedure is given by Algorithm 7.3 where ArcSetsGFCI is the version of Atamtürk [2002] for DIrected problems and EdgeSetsGFCI is a transformation for the BIdirected and UNdirected case.

Input : point $\bar{p} = (\bar{f}, \bar{x})$, node set $S \subset V$, commodity subset $Q \subseteq K$, facility $s \in T$

Output : arc sets $A_1^+ \subseteq A_S^+$ and $A_2^- \subseteq A_S^-$	**Output :** edge sets $E_1, E_2 \subseteq E_S$
1: $A_1^+ := \{\, a \in A_S^+ : \mathcal{G}_{d,c^s}(c^t)\bar{x}_a^t < \bar{f}_a^Q \,\}$	1: $E_1 := \{\, e = ij \in E_S : \mathcal{G}_{d,c^s}(c^t)\bar{x}_e^t < \bar{f}_{ij}^Q \,\}$
2: $A_2^- := \{\, a \in A_S^- :$ $(c^t + \mathcal{G}_{d,c^s}(-c^t))\bar{x}_a^t < \bar{f}_a^Q \,\}$	2: $E_2 := \{\, e = ij \in E_S :$ $(c^t + \mathcal{G}_{d,c^s}(-c^t))\bar{x}_e^t < \bar{f}_{ji}^Q \,\}$
3: **return** (A_1^+, A_2^-)	3: **return** (E_1, E_2)

Algorithm 7.3: ArcSetsGFCI($\bar{p}, S, Q, s$) EdgeSetsGFCI($\bar{p}, S, Q, s$)

These algorithms have a running time in $O(|A_S||T|)$ respectively $O(|E_S||T|)$ since the *MIR* coefficients $\mathcal{G}_{d,c^s}(c^t)$ and $(c^t + \mathcal{G}_{d,c^s}(-c^t))$ are evaluated in constant time.

Lemma 7.1 *Given a point $\bar{p} = (\bar{f}, \bar{x})$, a node set $S \neq \emptyset$, a commodity subset $Q \neq \emptyset$ and a facility $s \in T$, Algorithm 7.3 calculates subsets A_1^+ and A_2^- of the arcs A_S^+ and A_S^- (subsets E_1 and E_2 of the edges E_S) that give a most violated (general) flow cut inequality* (7.1) *(or* (7.2)*).*

Proof. Since the DIrected case is a result of Atamtürk [2002] we only give a proof for BIdirected and UNdirected problems here. Suppose E_1 and E_2 are the two subsets chosen with Algorithm 7.3. We assume that the statement is not true, so E_1 and E_2 do not give a most violated flow cut inequality. Hence there exist subsets E_3 and E_4 of E_S which lead to a a smaller left hand side value when evaluating the corresponding flow cut inequality:

$$\begin{aligned} & \bar{f}^Q(\bar{E}_3^+) - \bar{f}^Q(E_4^-) + \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t)\bar{x}^t(E_3) + \sum_{t\in T}(c^t + \mathcal{G}_{d,c^s}(-c^t))\bar{x}^t(E_4) \\ < \quad & \bar{f}^Q(\bar{E}_1^+) - \bar{f}^Q(E_2^-) + \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t)\bar{x}^t(E_1) + \sum_{t\in T}(c^t + \mathcal{G}_{d,c^s}(-c^t))\bar{x}^t(E_2). \end{aligned}$$

Now we show that by (resorting edges and) switching from E_3 to E_1 and from E_4 to E_2 we can only make the left hand side smaller which contradicts the assumption.

We use the fact that $E_3 = (E_1 \backslash (\bar{E}_3 \cap E_1)) \cup (E_3 \cap \bar{E}_1)$ and $E_4 = (E_2 \backslash (\bar{E}_4 \cap E_2)) \cup (E_4 \cap \bar{E}_2)$:

$$\begin{aligned} & \bar{f}^Q(\bar{E}_3^+) - \bar{f}^Q(E_4^-) + \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t)\bar{x}^t(E_3) + \sum_{t\in T}(c^t + \mathcal{G}_{d,c^s}(-c^t))\bar{x}^t(E_4) \\ = \quad & \bar{f}^Q(\bar{E}_1^+) - \bar{f}^Q(E_2^-) + \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t)\bar{x}^t(E_1) + \sum_{t\in T}(c^t + \mathcal{G}_{d,c^s}(-c^t))\bar{x}^t(E_2) \end{aligned}$$

$$
\begin{aligned}
&-\bar{f}^Q(E_3^+ \cap \bar{E}_1^+) \ + \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t)\bar{x}^t(E_3 \cap \bar{E}_1)\\
&+\bar{f}^Q(\bar{E}_3^+ \cap E_1^+) \ - \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t)\bar{x}^t(\bar{E}_3 \cap E_1)\\
&+\bar{f}^Q(\bar{E}_4^- \cap E_2^-) \ - \sum_{t\in T}(c^t + \mathcal{G}_{d,c^s}(-c^t))\bar{x}^t(\bar{E}_4 \cap E_2)\\
&-\bar{f}^Q(E_4^- \cap \bar{E}_2^-) \ + \sum_{t\in T}(c^t + \mathcal{G}_{d,c^s}(-c^t))\bar{x}^t(E_4 \cap \bar{E}_2)\\
\geq \ \ & \bar{f}^Q(\bar{E}_1^+) - \bar{f}^Q(E_2^-) + \sum_{t\in T} \mathcal{G}_{d,c^s}(c^t)\bar{x}^t(E_1) + \sum_{t\in T}(c^t + \mathcal{G}_{d,c^s}(-c^t))\bar{x}^t(E_2).
\end{aligned}
$$

The $=$ is simply rewriting and the $\geq$ follows from the choice of E_1 and E_2 in our procedure and is a contradiction to the assumption that E_1 and E_2 do not give a most violated inequality. This completes the proof. ■

To find a most violated strengthened simple flow cut inequality we have to apply a different separation procedure as the one for general flow cut inequalities above (see Algorithm 7.4). The running time of these procedures is again in $O(|A_S||T|)$ (resp. $O(|E_S||T|)$). To prove that they yield a most violated strengthened simple flow cut inequality, simply modify the proof of Lemma 7.1

Lemma 7.2 *Given a point $\bar{p} = (\bar{f}, \bar{x})$, a node set $S \neq \emptyset$, a commodity subset $Q \neq \emptyset$ and a facility $s \in T$, Algorithm 7.4 calculates a subsets A_1^+ of the arcs A_S^+ (a subset E_1 of the edges E_S) that gives a most violated simple flow cut inequality* (7.3) *(*(7.4)*).*

Input : point $\bar{p} = (\bar{f}, \bar{x})$, node set $S \subset V$, commodity subset $Q \subseteq K$, facility $s \in T$

Output : arc set $A_1^+ \subseteq A_S^+$	**Output :** edge sets $E_1 \subseteq E_S$
1: **return** $A_1^+ := \{\, a \in A_S^+ :$ $\min(r_s^Q \eta_s^Q, \mathcal{G}_{d,c^s}(c^t)\bar{x}_a^t) \ < \ \bar{f}_a^Q \,\}$	1: **return** $E_1 := \{\, e = ij \in E_S :$ $\min(r_s^Q \eta_s^Q, \mathcal{G}_{d,c^s}(c^t)\bar{x}_e^t) \ < \ \bar{f}_{ij}^Q \,\}$

Algorithm 7.4: ArcSetSFCI($\bar{p}$, S, Q, s) EdgeSetSFCI($\bar{p}$, S, Q, s)

Eventually consider the new cut set inequality (7.5) for single-facility BIdirected or UNdirected problems. For a fixed cut of the network defined by $S \subset V$ exact separation is done by Algorithm 7.5 in $O(|E_S|)$-time.

Input : point $\bar{p} = (\bar{f}, \bar{x})$, node set $S \subset V$

Output : an edge set $E_1 \subseteq E_S$

1: **return** $E_1 := \{\, e = ij \in E_S :\ r^{K^+}\bar{x}_e \ < \ \bar{f}_{ij}^{K^+} - \bar{f}_{ji}^{K^+} \,\}$

Algorithm 7.5: EdgeSetNCSI($\bar{p}$, S)

Lemma 7.3 *Given a point $\bar{p} = (\bar{f}, \bar{x})$ and a node set $S \neq \emptyset$, Algorithm 7.5 calculates a subset E_1 that gives a most violated cut set inequality* (7.5).

Proof. Assume that E_1 is chosen with the described procedure and suppose it does not give a most violated inequality. Hence there exists $E_2 \subseteq E_S$ with smaller left hand side:

$$\begin{aligned} & c\bar{x}(E_2) + \big(c - r^{K^+}\big)\bar{x}(\bar{E}_2) + \bar{f}^{K^+}(E_2^-) - \bar{f}^{K^+}(E_2^+) \\ < \quad & c\bar{x}(E_1) + \big(c - r^{K^+}\big)\bar{x}(\bar{E}_1) + \bar{f}^{K^+}(E_1^-) - \bar{f}^{K^+}(E_1^+) \end{aligned}$$

But with $E_2 = (E_1 \backslash (\bar{E}_2 \cap E_1)) \cup (E_2 \cap \bar{E}_1)$ it follows

$$\begin{aligned} & c\bar{x}(E_2) + \big(c - r^{K^+}\big)\bar{x}(\bar{E}_2) + \bar{f}^{K^+}(E_2^-) - \bar{f}^{K^+}(E_2^+) \\ = \quad & c\bar{x}(E_1) + \big(c - r^{K^+}\big)\bar{x}(\bar{E}_1) + \bar{f}^{K^+}(E_1^-) - \bar{f}^{K^+}(E_1^+) \\ & + c\bar{x}(E_2 \cap \bar{E}_1) \qquad\qquad - c\bar{x}(\bar{E}_2 \cap E_1) \\ & + \big(c - r^{K^+}\big)\bar{x}(\bar{E}_2 \cap E_1) \ - \big(c - r^{K^+}\big)\bar{x}(E_2 \cap \bar{E}_1) \\ & + \bar{f}^{K^+}(E_2^- \cap \bar{E}_1^-) \qquad\quad - \bar{f}^{K^+}(\bar{E}_2^- \cap E_1^-) \\ & - \bar{f}^{K^+}(E_2^+ \cap \bar{E}_1^+) \qquad\quad + \bar{f}^{K^+}(\bar{E}_2^+ \cap E_1^+) \\ = \quad & c\bar{x}(E_1) + \big(c - r^{K^+}\big)\bar{x}(\bar{E}_1) + \bar{f}^{K^+}(E_1^-) - \bar{f}^{K^+}(E_1^+) \\ & + r^{K^+}\bar{x}(E_2 \cap \bar{E}_1) \qquad\quad - r^{K^+}\bar{x}(\bar{E}_2 \cap E_1) \\ & + \bar{f}^{K^+}(E_2^- \cap \bar{E}_1^-) \qquad\quad - \bar{f}^{K^+}(\bar{E}_2^- \cap E_1^-) \\ & - \bar{f}^{K^+}(E_2^+ \cap \bar{E}_1^+) \qquad\quad + \bar{f}^{K^+}(\bar{E}_2^+ \cap E_1^+) \\ \geq \quad & c\bar{x}(E_1) + \big(c - r^{K^+}\big)\bar{x}(\bar{E}_1) + \bar{f}^{K^+}(E_1^-) - \bar{f}^{K^+}(E_1^+). \end{aligned}$$

The first $=$ is rewriting while the final $\geq$ follows from the way we have chosen E_1 and is a contradiction to the assumption that E_1 does not give a most violated inequality. The proof is complete. ∎

The three parts of the decomposition are now integrated into a separation procedure. The procedure is given by Algorithm 7.6 for BIdirected and UNdirected problems. The procedure for DIrected problems is similar except for the steps 5 to 8, the separation of the cut set inequality (7.5), which is omitted. Note that in step 14 we concentrate on (general) flow cut inequalities that are not simple because we consider strengthened simple flow cut inequalities separately. For every combination of cuts and commodity subsets, which are determined heuristically, and for every facility the algorithm computes a most violated general flow cut inequality (not simple) and a most violated simple flow cut inequality. If $T = 1$ a most violated cut set inequality (7.5) is calculated in addition. All these inequalities are put into a pool, which finally can be quite large. The next section tries to answer the question, how to handle a large amount of violated inequalities effectively and how to integrate the separation procedure into a Branch & Cut algorithm. The running time for the separation heuristic Algorithm 7.6 can roughly be estimated by

$$O\big(|K|\big(|T|^2|A|\log|A| + |V|\big)\big) \quad \text{respectively} \quad O\big(|K|\big(|T|^2|E|\log|E| + |V|\big)\big).$$

Note that the number of node sets calculated in step 2 does not depend on the instance but on the size of the final partition *PSize*. But since the number of node sets is exponential in *PSize* the implicit factor in the estimate can be quite large. For UNdirected problems one has to ensure that $K^- = \emptyset$ in step 4. Given a cut E_S, this can be done by swapping flow variables for negative commodities as described in Section 4.1.

Input : point $\bar{p} = (\bar{f}, \bar{x})$, *PSize* ≥ 2, $G = (V, E)$, net demands d_i^k, $i \in V, k \in K$
facilities T, capacities c^t, $t \in T$

Output : a pool $CutPool$ of flow cut inequalities (7.2), simple flow cut inequalities (7.4) and cut set inequalities (7.5) all violated by $\bar{p}$

Initialise $CutPool := \emptyset$.
$L_S =$ NODESETSBYCONTRACTION($\bar{p}$, *PSize*, G)
for $S \in L_S$ **do**
  For UNdirected problems ensure that $K^- = \emptyset$ (by swapping flow variables).
  if $|T| = 1$ **then**
    E_1 = EDGESETSNCSI($\bar{p}$, S)
    Calculate the unique cut set inequality (7.5) w.r.t S and E_1, if violated put it to the $CutPool$.
  end if
  $L_Q =$ COMMODITYSUBSETS(S, d)
  for $Q \in L_Q$ **do**
    If $Q \subseteq K^-$ then set $S := V \backslash S$. (Ensure to consider positive commodity subsets.)
    for $s \in T$ **do**
      (E_1, E_2) = EDGESETSGFCI($\bar{p}$, S, Q, s)
      if $E_2 \neq \emptyset$ **then**
        Calculate the unique flow cut inequality (7.2) w.r.t S, Q, E_1, E_2 and s, if violated put it to the $CutPool$.
      end if
      E_1 = EDGESETSFCI($\bar{p}$, S, Q, s)
      Calculate the unique simple flow cut inequality (7.4) w.r.t S, Q, E_1 and s, if violated put it to the $CutPool$.
    end for
  end for
end for
return $CutPool$

Algorithm 7.6: SEPARATIONHEURISTICFLOWCUT($\bar{p}$, *PSize*, G, d, T, c)

Algorithm 7.6 only provides a general framework. It will be modified and extended in the next section.

Summary We have defined and discussed the separation problem for general flow cut inequalities, for simple flow cut inequalities and for the new cut set inequalities (7.5).

The problem has been decomposed and a fast separation procedure has been proposed that exploits the value of the point $\bar{p}$ and that is able to calculate a pool of inequalities all violated by $\bar{p}$.

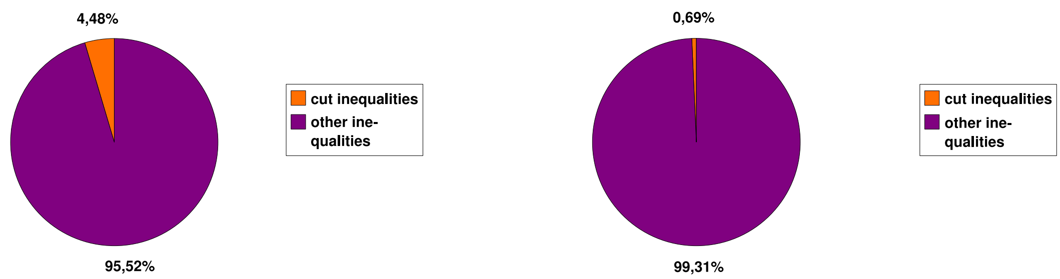
7.3 Implementational aspects

In the following it will be discussed how to integrate a separation heuristic such as Algorithm 7.6 into a state-of-the-art Branch & Cut framework. We used CPLEX 9.0 (ILOG [2005]) for the implementation. CPLEX applies a bunch of separators, heuristics and branching rules within a Branch & Bound framework to solve general mixed integer programs. We will see that this generic approach together with the cutting planes specific for network design problems provides a powerful tool to optimise even larger networks.

CPLEX's default settings were not changed and we made use of **callbacks** to integrate our separation heuristic into the Branch & Cut algorithm. A user-written cut-callback is called at each node of the Branch & Bound tree having an LP optimal solution $\bar{p}$ that is fractional and that has an objective below the best known upper bound. The callback may add globally valid inequalities to the initial formulation, that are violated by $\bar{p}$. These inequalities remain part of all subsequent sub-problems and apply throughout the Branch & Bound tree. There is no cut deletion (ILOG [2005]).

Modifications and improvements of the separation heuristic In a first attempt Algorithm 7.6 was implemented as a callback without any modifications and all violated inequalities were added to the formulation. We were interested in the behaviour of the separation heuristic within CPLEX's Branch & Cut framework. The results of the initial experiments were disappointing. We will not state them in detail but will summarise the main drawbacks of a naive implementation of Algorithm 7.6 in the following:

- Although cut inequalities form a subclass of all three considered classes of inequalities they are seldomly added to the $CutPool$. Given a cut of the network and one of the sets K^+ or K^-, cut inequalities almost never belong to the most violated inequalities calculated by Algorithm 7.6. The absence of strong cut inequalities really reduces the performance. Figure 7.1 shows a usual distribution of the most violated inequalities and the inequalities that are most violated and in fact violated by $\bar{p}$ considered by Algorithm 7.6.



(i) $|T| > 1$: 4915712 most violated inequalities (ii) $|T| > 1$: 500373 of 4915712 inequalities violated by $\bar{p}$

Figure 7.1: Distributions of (most) violated inequalities calculated by SEPARATIONHEURISTICFLOWCUT The statistics were made by testing Algorithm 7.6 as a callback against 10 instances all modelled BIdirected and a time limit of 30 minutes.

- The violation of an inequality seems not to be a good measure for its quality, i. e., its ability to increase the lower bound.

- The number of violated inequalities found by the separator is enormous for most of the instances. Adding them all leads to large LP-relaxations and unacceptable computation times for solving them.

- The same inequality might be added to the $CutPool$ several times and many inequalities in the pool are almost parallel (with an orthogonality close to zero).

As a first improvement it was decided to implement a second separation heuristic that solely computes cut inequalities. Given a cut A_S and a facility $s \in T$, we consider the two (strengthened) cut inequalities

$$\sum_{t \in T} \min\left(r_s^{K+} \eta_s^{K+}, \mathcal{G}_{d,c^s}(c^t)\right) x^t(A_S^+) \geq r_s^{K+} \eta_s^{K+} \qquad \text{and} \tag{7.6}$$

$$\sum_{t \in T} \min\left(r_s^{K-} \eta_s^{K-}, \mathcal{G}_{d,c^s}(c^t)\right) x^t(A_S^-) \geq r_s^{K-} \eta_s^{K-} \tag{7.7}$$

for DIrected problems with $r_s^{K+}\eta_s^{K+} = r(d_S^{K+}, c^s)\lceil \frac{d_S^{K+}}{c^s} \rceil$ and $r_s^{K-}\eta_s^{K-} = r(|d_S^{K-}|, c^s)\lceil \frac{|d_S^{K-}|}{c^s} \rceil$. For BIdirected and UNdirected problems, a cut E_S and a facility $s \in T$ we consider the (strengthened) cut inequality

$$\sum_{t \in T} \min\left(r(d, c^s)\lceil \tfrac{d}{c^s} \rceil, \mathcal{G}_{d,c^s}(c^t)\right) x^t(E_S) \geq r(d, c^s)\lceil \tfrac{d}{c^s} \rceil, \tag{7.8}$$

where $d := \max(d_S^{K+}, |d_S^{K-}|)$. Algorithm 7.7 uses the same contraction procedure as Algorithm 7.6 and checks all cut inequalities corresponding to the cuts of the shrunken graph for violation.

Input : point $\bar{p} = (\bar{f}, \bar{x})$, *PSize* ≥ 2, $G = (V, E)$ or $G = (V, A)$,
net demands d_i^k, $i \in V, k \in K$ facilities T, capacities c^t, $t \in T$
Output : a pool $CutPool$ of cut inequalities all violated by $\bar{p}$

```
Initialise CutPool := ∅.
L_S = NodeSetsByContraction(p̄, PSize, G)
for S ∈ L_S do
    Calculate K+ and K− with respect to S.
    For UNdirected problems ensure that K− = ∅ (by swapping flow variables).
    for s ∈ T do
        DIrected: If (7.6) is violated by p̄ add it to the CutPool.
        DIrected: If (7.7) is violated by p̄ add it to the CutPool.
        BIdirected, UNdirected: If (7.8) is violated by p̄ add it to the CutPool.
    end for
end for
return CutPool
```

Algorithm 7.7: SeparationHeuristicCut($\bar{p}$, *PSize*, G, d, T, c)

We are now faced with two different separation heuristics that can be used independently from each other. To further improve the overall performance some additional modifications have been implemented. To determine the standard settings for lots of parameters, a series of tests has been done for all instances. We cannot give a detailed parameter discussion here. We only sketch some of the improvements and motivate some settings:

- Only add violated inequalities to the $CutPool$ if they are not too parallel to inequalities already in the pool and to inequalities already in the formulation.

- Introduce a measure for the efficiency of a violated inequality. We used a linear combination of the distance to the fractional point $\bar{p}$ and the parallelism with respect to the hyperplane given by the objective function.

  Add a violated inequality to the pool only if its efficiency is greater than a certain minimum. The minimal efficiency should be changed dynamically. Increase it if there are too many inequalities added and decrease it if the pool is (almost) empty.

  After the termination of the separation heuristics the pool gets sorted with respect to the efficiency and only a small number of the best inequalities (with large distance to $\bar{p}$ and almost parallel to the objective) in the pool is added to the formulation.

- Limit the total number of separated inequalities. If m is the number of rows of the initial formulation, we not allowed the callbacks to add more than m cutting planes during the whole optimisation process.

- Do not apply the separation heuristics at every node of the Branch & Bound tree but only in certain depths.

- The separation of (simple) flow cut inequalities and cut set inequalities of type (7.5) should be done carefully. It turned out that it is useful to apply the modules Algorithm 7.7 and Algorithm 7.6 in a hierarchical manner. We only executed the module Algorithm 7.6 if in a certain number of iterations there were no violated cut inequalities.

- The size of the shrunken graph calculated by Algorithm 7.1 should be small, $2 \leq \textit{PSize} \leq 5$. We fixed $\textit{PSize} := 3$ and the module Algorithm 7.7 additionally checked all multi cut inequalities (6.1) or (6.3) and added them to the pool if violated.

With these modifications we are now prepared for the final tests of the efficiency of our separation heuristics.

7.4 Computational results

7.4.1 Data sets

For our tests we selected instances from the *SNDlib 1.0 – Survivable Network Design Data Library* [2005], which has been launched recently by R. Wessäly and M. Pióro and contains realistic data sets for (survivable) telecommunication network design. Table 7.1 states all used problems.

Each instance is given by a supply graph (nodes V and links E), a set of (directed) demands D and a set of installable link designs T. Demands were aggregated to obtain a set of commodities K as described in Section 2.2.1. It holds for all selected instances that every technology is installable on all links of the network. To model BIdirected and UNdirected problems as considered in this thesis, every link was interpreted as being undirected. For DIrected problems it was assumed that for every link between nodes u and v there is an directed arc (u, v) and an directed arc (v, u), such that the

problem	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert D\rvert$	$\lvert K\rvert$	$\lvert T\rvert$
di-yuan	11	42	22	8	7
newyork	16	49	240	16	2
zib16	16	51	77	15	5
nobel-germany	17	26	121	15	40
france	25	45	300	25	1
norway	27	51	702	27	2
sun	27	102	67	18	1
nobel-eu	28	41	378	27	40
pioro40	40	89	780	39	2
zib54	54	81	1501	42	1
ta2	65	108	1869	42	11

Table 7.1: Data set from *SNDlib 1.0 – Survivable Network Design Data Library* [2005]

total number of arcs $|A|$ equals $2|E|$. Every given pre-installed capacity was removed. We considered modular link capacities as well as explicit link capacities. There is no flow cost.

All calculations were done on a 2×3 GHz machine with 4 MB of memory. The computational results are presented in Appendix A in detail. Table A.1, Table A.2 and Table A.3 report results for modular link capacities and the problem types DIrected, BIdirected and UNdirected respectively. Table A.4, Table A.5 and Table A.6 show the efficiency of the considered separation heuristics in the presence of additional GUB constraints and hence models with explicit link capacities.

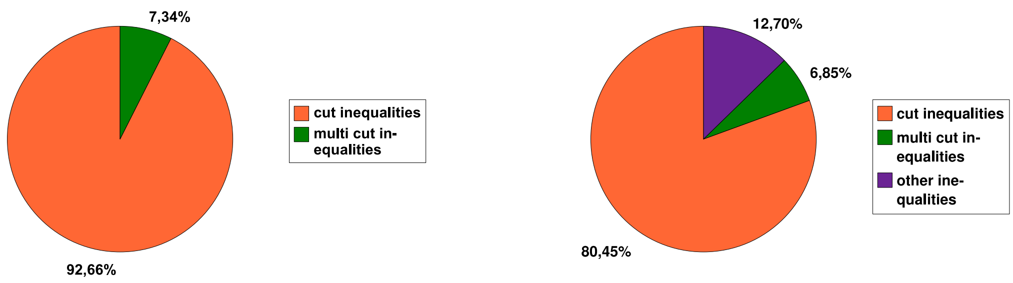
Every problem was tested with CPLEX and no callbacks, with CPLEX and the separation of cut as well as multi cut inequalities and eventually with an implementation that used both separation heuristics Algorithm 7.7 and Algorithm 7.6.

In the following we will analyse these results and state some statistics. First note that comparing the performance of different test cycles is difficult and hast to be done carefully. Every single violated inequality we add to the initial formulation influences various sub-algorithms that are used by CPLEX within the Branch & Cut framework such as heuristics, general purpose separators and branching rules. Note that we do not test our separation heuristics within a pure Branch & Bound algorithm but within (and against) a sophisticated state- of-the-art MIP-solver. Nevertheless, the results may serve as an indicator of the efficiency of our separation heuristics.

7.4.2 Results for modular link capacities

After the implementation of all mentioned modifications to the first approach, the distribution of inequalities that are added to the initial formulation has changed drastically. Since the separation of cut inequalities is now considered separately, these inequalities now dominate the overall separation process. This can be seen in Figure 7.2 when compared to Figure 7.1. This change of the distribution has made the solution process much faster and robust and can be seen as the major reason for the excellent results that have been obtained.

Before giving an overall statistic we briefly summarise the results reported by Table A.1, Table A.2 and Table A.3 for modular link capacities. The most important observations are the following:



(i) Algorithm 7.7 only (ii) Both separation heuristics

Figure 7.2: Distribution of separated inequalities – modular capacities

- For all data sets and all problem types we could reduce the computation time or the final gap.
- By only applying the separation heuristic Algorithm 7.7 and separating cut as well as multi cut inequality the acceleration of the optimisation process is enormous.
- Additionally separating (simple) flow cut inequalities by Algorithm 7.6 still results in an improvement of the overall performance but this improvement is small.
- The separation heuristics behave robust and stable. The added inequalities have integer coefficients that are small with respect to the given capacities, which follows from Corollary 3.8. We reported numerical problems only for a single test instance due to ill posed data (see below).

The behaviour of the separation heuristics was independent from the problem type. This is due to the fact that the heuristics pay regard to the different structure of the models. In the following we will not distinguish them anymore but consider all instances at once.

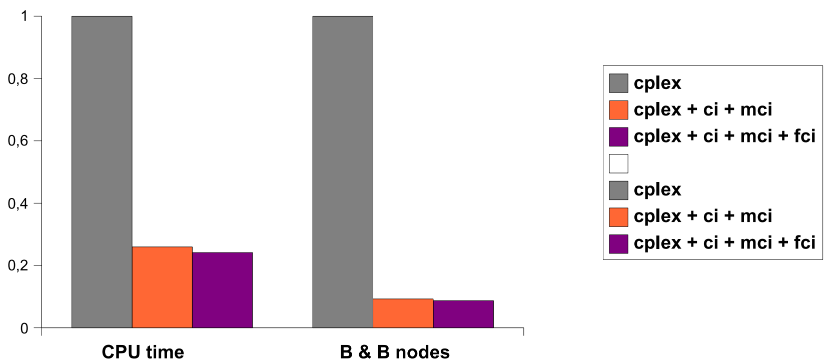


Figure 7.3: Easy to solve test instances and improvements made by adding separation heuristics to CPLEX's default MIP-solver – modular link capacities

There are 11 data sets and 3 problem types. From the 33 resulting instances 3 could be solved within the time limit of 1 hour by CPLEX independently from adding separators or not. For all

3 instances we reduced the computation time as well as the visited Branch & Bound nodes when adding our separation heuristics. For another 8 instances CPLEX ended up with a gap between 2% and 8% but the problems were solved when applying our separators. The solution time was even under 10 minutes for most of them.

We were obviously able to add all the necessary strong cutting planes for these 11 small to medium sized examples. Figure 7.3 provides a statistic for them comparing the solution time and the visited nodes during Branch & Bound for adding cut and multi cut inequalities (ci + mci), for adding all considered cutting planes (ci + mci + fci) in ratio to the values obtained by CPLEX without callbacks (cplex). If CPLEX ended up with a gap we considered a solution time of 1 hour and the nodes so far visited. Hence the actual improvement is even greater than shown by Figure 7.3.

For the remaining 22 instances that could not be solved to optimality the endgaps were significantly better when applying the separators and for 20 of them we improved the best solution. The final gap could be reduced for 13 of those 22 instances by more than 50%. Even large instances as zib54 ended up with a gap under 15%. Figure 7.4 reports the improvements made in the lower and upper bound, the final gap and the number of nodes in the search tree again in ratio to the values obtained by CPLEX's default MIP-solver. We can provide better solutions and better quality certificates and need to explore less Branch & Bound nodes for it.

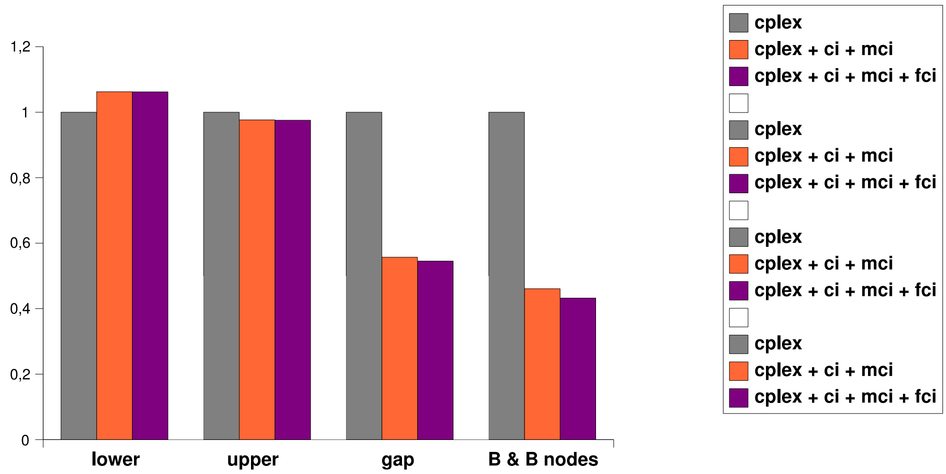


Figure 7.4: Hard to solve test instances and improvements made by adding separation heuristics to CPLEX's default MIP-solver – modular link capacities

Both charts show that there is a great progress when only adding cut and multi cut inequalities. The performance is still perceivable better when additionally separating (simple) flow cut inequalities but it is only a small improvement.

We had some problems with three of the instances. Although the CPU time consumed by the separation heuristics can be neglected for the rest of instances, for nobel-germany and nobel-eu it is simply too large. These two examples have a large number of facilities. Our separation heuristics try to detect violated cutting planes for every given base capacity in each iteration. More research is needed to exploit the structure of the given capacity values. Nevertheless, even here we reduce the overall gap or solve these instances within the time limit, whereas CPLEX runs into problems. In the DIrected case (the most difficult of the three problem types, since we doubled links and hence design variables) CPLEX terminates because of a reached memory limit for both nobel-germany and nobel-eu. The results for the largest problem, ta2 with DIrected capacity usage, might be incorrect. The simplex algorithm terminated late in the optimisation process because of a singular basis, which indicates numerical problems. Note that the capacities c^t for ta2 are given in a magnitude of 10^7 while the default precision of CPLEX is 10^{-6} and all our *MIR* inequalities are obtained by dividing base inequalities by the values c^t. One has to rescale the data or to increase the precision of the calculations.

Let us finally try to answer the question why applying Algorithm 7.6 in addition to the separation of cut and multi cut inequalities results only in slight improvements of the performance and how this can be fixed.

First the number of added inequalities by Algorithm 7.6 is small but tests showed that increasing the number of (simple) flow cut inequalities leads to unacceptable overall computation times. Flow cut inequalities seem to be somewhat weaker than cut and multi cut inequalities and it is rather a problem to find strong valid inequalities than to find violated ones. It was already mentioned that except for nobel-eu and nobel-germany the computation time consumed by our separation heuristics is very small. Hence one could spend more time to solve the separation problem.

The author believes that the biggest drawback of Algorithm 7.6 is the simple heuristic Algorithm 7.2 to find promising commodity subsets. Important strong valid flow cut inequalities might be missed when largely concentrating on single commodities. But there is no better heuristic known so far.

Our approach of finding promising cuts is based on contraction of the network, which is a very fast heuristic. But especially for sparse networks we might end up with cuts E_S (or A_S) that do not give strong valid flow cut inequalities. Again remember from Section 4.1 that a cut set inequality is facet defining for a network design polyhedron if it defines a facet for the corresponding cut set and if $G[S]$ and $G[V\backslash S]$ are connected (undirected graphs) or strongly connected (directed graphs). Initially calculating cuts with this property was successfully used by Bienstock et al. [1995] and Bienstock & Günlük [1996]. A second possible approach is to exploit the structure of the components of $G[S]$ and $G[V\backslash S]$ in order to strengthen cut set inequalities. For sparse networks more research has to be done (see Section 6.6).

Eventually the author conjectures that flow cut inequalities behave better in the presence of flow cost because then they are in some sense more parallel to the objective function.

7.4.3 Results for explicit link capacities

The separation heuristics were implemented and tested against different parameter settings with respect to the models considered in this thesis. These are given by modular link capacities. Without changing the algorithms and without doing new parameter tests we were interested in the behaviour of the investigated inequalities in the presence of GUB constraints

$$\sum_{t\in T} x_a^t \leq 1 \ \forall a \in A, \qquad \sum_{t\in T} x_e^t \leq 1 \ \forall e \in E.$$

Notice that GUB constraints imply the bound constraints $x_a^t \leq 1$ for all $a \in A, t \in T$ (or $x_e^t \leq 1$ for all $e \in E, t \in T$). It follows that the unbounded and bounded network design problems defined in

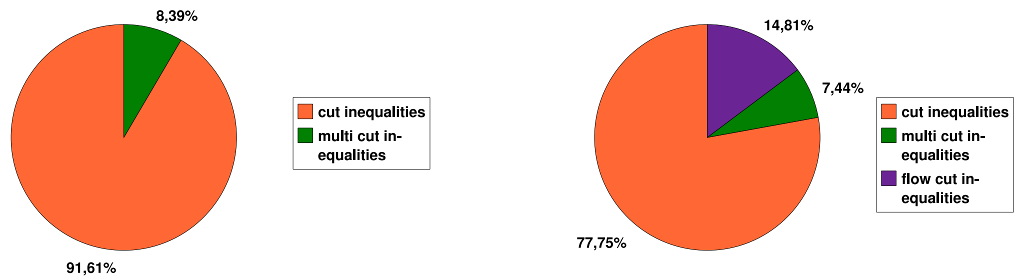
Chapter 2 and studied throughout the thesis are relaxations of problems with explicit link capacities and hence all developed inequalities are still valid and can be used to tighten the initial formulation.

One would expect them to be weak but as the following results show they are of significant practical usefulness even for models with GUB constraints.

Not all of the data sets can be used to model explicit capacities. The limitation of the possible amount of capacity that can be installed does not allow for a feasible routing for some of the instances. The corresponding polyhedra are empty. This is the case for the data sets

- france, norway, newyork and pioro40

A first observation is, that with a total number of separated inequalities that is almost the same, the distribution changes slightly compared to the tests for modular link capacities (see Figure 7.5 compared to Figure 7.2). There are less cut inequalities and more multi cut inequalities and (simple)



(i) Algorithm 7.7 only

(ii) Both separation heuristics

Figure 7.5: Distribution of separated inequalities – explicit link capacities

flow cut inequalities added to the initial formulation. Detailed results can be found in Table A.4, Table A.5 and Table A.6.

From the left 21 problems, di-yuan and zib16 in the BIdirected case are solved more quickly with CPLEX and no callbacks. For another two examples CPLEX ends up with a better gap. These are nobel-eu in the DIrected and nobel-eu in the BIdirected case. Note that this is not due to a better lower bound but because of better solutions. For the rest of the instances adding the separation heuristics to CPLEX's default Branch & Cut algorithm results in overall improvements of the performance, in some cases enormous. So for 11 instances CPLEX ends up with a gap and by adding the heuristics we solve these instances to optimality or reduce the gap clearly by more than 50%.

Figure 7.6 and Figure 7.7 again report overall statistics for easy (solved to optimality within 1 hour of computation time) and hard to solve instances. The values are given in ration to the values obtained by CPLEX without any callbacks. When comparing these charts to Figure 7.3 and Figure 7.4 it can be seen that the reduction of computation time and gap for models with explicit link capacities is still significant but smaller than for models with modular link capacities.

It is interesting and unexpected that the effect of adding (simple) flow cut inequalities and cut set inequalities of type (7.5) in addition to cut and multi cut inequalities is much greater for models with explicit capacities. This might be due to the change of the distribution of the separated inequalities. It happens more often that the Algorithm 7.7 fails to find violated cut inequalities such that Algorithm 7.6 is applied more often and seems to help out.

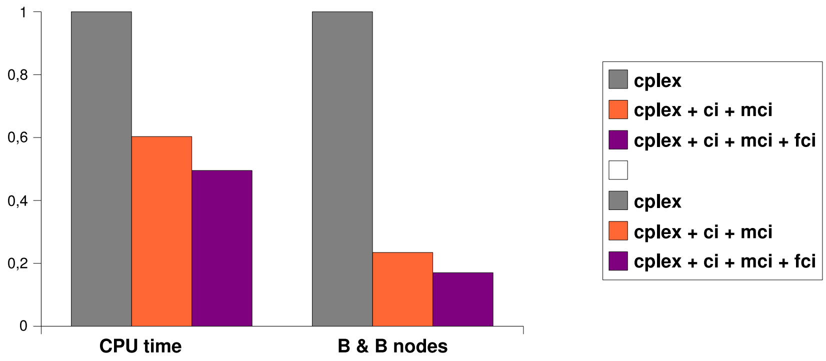


Figure 7.6: Easy to solve test instances and improvements made by adding separation heuristics to CPLEX's default MIP-solver – explicit link capacities

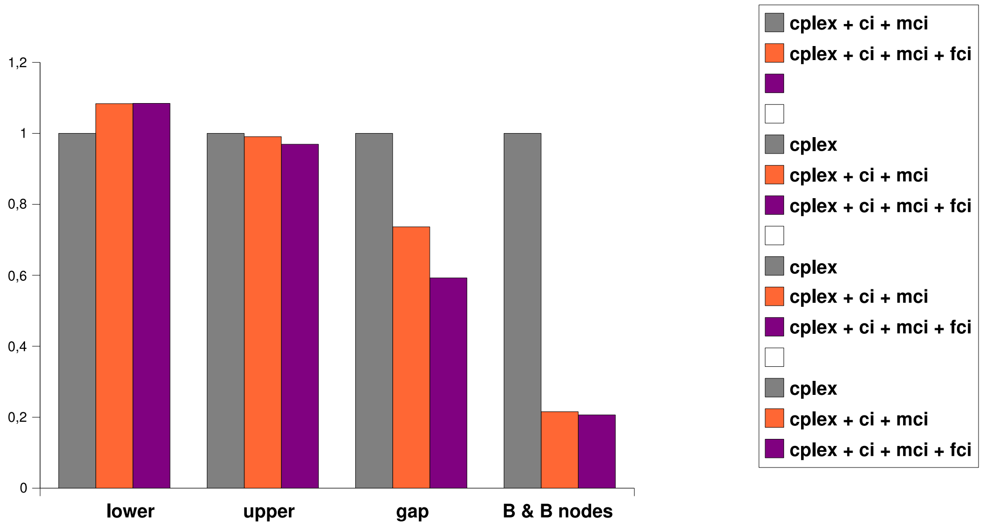


Figure 7.7: Hard to solve test instances and improvements made by adding separation heuristics to CPLEX's default MIP-solver – explicit link capacities

7.5 Summary

We considered the separation problem for the strong valid cut set inequalities of Chapter 4 and developed separation heuristics that can be used within a state-of-the-art Branch & Cut MIP-solver such as CPLEX. These algorithms were tested against real-world networks.

It turned out that it is useful to separately apply a heuristic that concentrates on cut inequalities and multi cut inequalities as considered in this thesis and only to add (simple) flow cut inequalities if no such cutting planes can be found. The improvements in the overall solution time and the final gap that could be made are enormous compared to CPLEX's results obtained without any callbacks.

This is even true for models with explicit link capacities, although GUB or upper bound constraints were not considered when developing the inequalities that were integrated into the separation modules. It can be conjectured that the optimisation process can still be accelerated when additionally separating inequalities that explicitly exploit bound or GUB constraints as for instance the flow cover inequalities of Chapter 5. Moreover, the stated algorithms can be extended easily for the separation of flow cover inequalities. We can use the basic framework and additionally find flow covers and complement variables (see Chapter 5).

Chapter 8

Conclusion

Telecommunication network design problems with bifurcated routing and a finite set of installable base capacities have been studied. We have considered bounded design variables as well as unbounded design variables and distinguished three common types of capacity usage, DIrected, BIdirected and UNdirected. The corresponding models arise as parts and sub-problems of larger and more complex problems containing additional requirements such as survivability of the network, hop limits or costs for hardware installation at the nodes of the network. All problems have been formulated as mixed integer programs. A successful approach to solve such problems is the use of Branch & Bound methods in combination with cutting plane algorithms (Branch & Cut). For the effectiveness of these algorithms it is crucial to understand the facial structure of the considered polyhedra.

We focused on the use of *Mixed-Integer Rounding* to develop strong valid inequalities for network design polyhedra. It was shown that by exploiting the structure of the given parameters such as underlying graphs, capacities and bound constraints and applying a general *MIR* procedure consisting of the steps *Aggregating*, *Substituting*, *Complementing* and *Scaling*, it is possible to derive different classes of strong valid or even facet-defining inequalities. Moreover, the use of *MIR* as a valid superadditive lifting function was emphasised.

In Chapter 4 and Chapter 5 it has been concentrated on so-called *cut sets* or *single node flow sets*, relaxations of network design polyhedra that are defined with respect to cuts of the network. As a central result we stated sufficient conditions for facet-defining inequalities of cut sets to define facets for network design polyhedra. A general class of facet-defining inequalities obtained by *MIR*, so-called flow cut inequalities, has been introduced and it has been investigated in detail. In the presence of bound constraints we could even generalise this class to the well-known flow cover inequalities.

These cutting planes have been considered for the three problem types DIrected, BIdirected and UNdirected and structural differences of the corresponding polyhedra have been elaborated on.

As an extension, Chapter 6 motivates some extended *MIR* techniques and provides a survey of several classes of strong valid inequalities for network design polyhedra that can be obtained by the developed *MIR* procedure.

Eventually we sketched the separation problem for some of the investigated inequalities. Separation heuristics have been implemented in a Branch & Cut framework and they have been tested against a bunch of realistic network design problems. We could demonstrate the efficency of the considered cutting planes and hence the usefulness of *MIR* in the context of telecommunication network design.

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List of Algorithms

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Appendix A

Results

The tables that can be found on the following pages report detailed results of our final tests. For all instances these tables compare test cycles done with CPLEX and no additional callbacks (*none*), with Algorithm 7.7 implemented as a callback (*ci+mci*) calculating cut inequalities as well as multi cut inequalities and with both separation heuristics Algorithm 7.7 and Algorithm 7.6 (*all*).

The first two columns (*problem*) and (*sep*) state the problem and the applied separators (in addition to the general purpose separators of CPLEX). In the following three columns we report the objective value of the LP relaxation (*lp*), the final dual bound (*lower*) and the objective value of the best (mixed) integer solution found (*upper*). Note that for clarity we only state the first 4 (or 5) leading digits. The next column (*time/gap*) provides the final gap or it reports the CPU time (given in $\mathrm{mm} :: \mathrm{ss}$) elapsed if the instance could be solved to optimality within a time limit of 1 hour. Column *nodes* reports the total number of nodes explored in the search tree during Branch & Cut and column *time sep* gives the total CPU time needed by the separation heuristics. The last three columns report the number of cut inequalities (*nof ci*), the number of multi cut inequalities (*nof mci*) and finally the number of (simple) flow cut inequalities and cut set inequalities of type (7.5) (*nof fci*) that were added to the initial formulation.

A.1 Modular capacities

problem	*sep*	*lp*	*lower*	*upper*	*time / gap %*	*nodes*	*time sep*	*nof ci*	*nof mci*	*nof fci*
di-yuan	none	3161	7314	7776	5.93	426382				
	ci + mci	3161	7621	7621	11:11	28482	00:00	175	0	
	all	3161	7621	7621	07:11	28482	00:01	175	0	0
france	none	1887	2059	2200	6.40	401526				
	ci + mci	1887	2130	2200	3.14	159923	24	138	0	
	all	1887	2128	2180	2.36	134846	00:51	128	0	166
newyork	none	6385	7093	7485	5.24	199550				
	ci + mci	6385	7239	7446	2.78	78309	00:12	193	1	
	all	6385	7241	7446	2.74	63531	00:29	196	1	143
nobel-eu	none	8588	8863	12266	27.7**M**	247385				
	ci + mci	8588	9149	11371	19.5	115595	07:32	402	2	
	all	8588	9062	11315	19.9	82747	26:48	363	2	83
nobel-germany	none	1474	1589	2172	26.8**M**	956134				
	ci + mci	1474	1815	1843	1.48	367965	20:23	246	3	
	all	1474	1815	1894	4.16	256191	37:14	215	3	77
norway	none	1627	1635	1676	2.44	640842				
	ci + mci	1627	1647	1673	1.53	254499	00:57	205	0	
	all	1627	1647	1672	1.52	251512	02:27	200	0	15
pioro40	none	4120	4137	4203	1.57	80157				
	ci + mci	4120	4142	4205	1.50	63877	00:20	119	0	
	all	4120	4142	4205	1.50	57087	00:54	106	0	8
sun	none	744.8	831.6	1054	21.0	69475				
	ci + mci	744.8	933.7	1050	11.0	24175	00:06	234	3	
	all	744.8	932.9	1050	11.1	25319	00:14	241	3	102
ta2	none	1064	1382	3613	61.7	6548				
	ci + mci	1064	1454	3592	59.5	2995	00:02	480	25	
	all	1064	1454	3592	59.5	2955	00:04	460	21	0
zib16	none	2097	2954	3397	13.0	264511				
	ci + mci	2097	3011	3408	11.6	75224	00:02	343	0	
	all	2097	3013	3321	9.2	72515	00:08	329	0	14
zib54	none	3813	10703	15994	33.1	7338				
	ci + mci	3813	13693	14697	6.83	4282	00:01	257	20	
	all	3813	13695	14697	6.82	4341	00:02	258	20	9

Table A.1: Results DIrected modular link capacities

problem	*sep*	*lp*	*lower*	*upper*	*time/ gap*	*nodes*	*time sep*	*nof ci*	*nof mci*	*nof fci*
di-yuan	none	2746	5537	5537	02:26	23880				
	ci + mci	2746	5537	5537	02:35	10187	00:00	141	31	
	all	2746	5537	5537	02:28	10187	00:00	141	31	0
france	none	1096	1181	1240	4.689	391378				
	ci + mci	1096	1240	1240	02:06	3279	00:00	53	6	
	all	1096	1240	1240	02:05	3316	00:00	53	6	5
newyork	none	3312	3697	3796	2.593	188325				
	ci + mci	3312	3780	3780	29:42	42029	00:03	274	19	
	all	3312	3780	3780	27:28	36872	00:04	245	18	91
nobel-eu	none	6019	6120	6697	8.626	856138				
	ci + mci	6019	6317	6461	2.233	349235	07:50	222	12	
	all	6019	6318	6435	1.823	258124	28:09	273	11	38
nobel-germany	none	1154	1241	1336	7.083	2809373				
	ci + mci	1154	1325	1325	05:55	91242	01:52	158	13	
	all	1154	1325	1325	05:08	49161	04:40	140	11	37
norway	none	8483	8531	8627	1.110	410441				
	ci + mci	8483	8573	8606	0.384	335596	00:41	69	0	
	all	8483	8574	8605	0.366	331433	01:44	63	0	13
pioro40	none	2540	2545	2559	0.562	105351				
	ci + mci	2540	2545	2559	0.529	100843	00:20	22	0	
	all	2540	2545	2559	0.523	104105	00:53	21	0	7
sun	none	570.2	604.4	700.8	13.758	75456				
	ci + mci	570.2	648.7	697.4	6.987	14845	00:02	340	43	
	all	570.2	648.7	695.5	6.739	14292	00:05	339	47	79
ta2	none	6724	8548	17144	50.138	6796				
	ci + mci	6724	8546	16437	48.002	2341	00:03	1713	391	
	all	6724	8546	16437	48.002	2415	00:05	1713	391	0
zib16	none	1582	2182	2182	21:25	199260				
	ci + mci	1582	2182	2182	07:09	16886	00:01	201	19	
	all	1582	2182	2182	07:16	14242	00:04	187	19	198
zib54	none	2018	7719	9788	21.136	10603				
	ci + mci	2018	8697	10216	14.863	7335	00:02	1406	629	
	all	2018	8699	10216	14.847	7358	00:04	1406	629	0

Table A.2: Results BIdirected modular link capacities

problem	*sep*	*lp*	*lower*	*upper*	*time/ gap*	*nodes*	*time sep*	*nof ci*	*nof mci*	*nof fci*
di-yuan	none	3161	6557	6557	15:20	223903				
	ci + mci	3161	6557	6557	02:09	14877	00:00	117	13	
	all	3161	6557	6557	02:07	14877	00:00	117	13	0
france	none	1887	1979	2020	1.99	964435				
	ci + mci	1887	2020	2020	00:02	284	00:00	27	2	
	all	1887	2020	2020	00:02	284	00:00	27	2	0
newyork	none	6386	6716	6820	1.53	817467				
	ci + mci	6386	6790	6790	06:45	23160	00:01	234	7	
	all	6386	6790	6790	05:25	18343	00:04	221	11	35
nobel-eu	none	8588	8686	9190	5.49	1057431				
	ci + mci	8588	8962	8985	0.26	293300	07:09	295	18	
	all	8588	8958	9000	0.47	247342	30:14	279	20	6
nobel-germany	none	1474	1530	1664	8.05	2835350				
	ci + mci	1474	1619	1619	00:42	6467	00:12	142	22	
	all	1474	1619	1619	00:59	7060	00:54	150	22	2
norway	none	1627	1631	1644	0.78	1090799				
	ci + mci	1627	1638	1643	0.30	496723	01:17	105	0	
	all	1627	1638	1643	0.30	493550	03:21	113	0	11
pioro40	none	4120	4125	4145	0.49	177311				
	ci + mci	4120	4127	4144	0.41	140488	00:30	35	0	
	all	4120	4127	4144	0.41	139770	01:31	37	0	5
sun	none	744.8	776.0	867.3	10.5	204753				
	ci + mci	744.8	851.4	863.7	1.43	40902	00:05	229	20	
	all	744.8	851.4	863.7	1.43	33975	00:11	223	20	151
ta2	none	1064	1218	2183	44.2	15600				
	ci + mci	1064	1233	1970	37.4	2754	00:04	2047	441	
	all	1064	1233	1970	37.4	2754	00:08	2047	441	0
zib16	none	2097	2703	2786	2.98	784484				
	ci + mci	2097	2757	2757	23:16	63778	00:00	273	18	
	all	2097	2757	2757	21:55	63778	00:01	273	18	0
zib54	none	3813	7193	10687	32.7	23693				
	ci + mci	3813	9228	10334	10.7	7442	00:02	1298	539	
	all	3813	9228	10334	10.7	7491	00:04	1300	540	0

Table A.3: Results UNdirected modular link capacities

A.2 Explicit capacities

problem	*sep*	*lp*	*lower*	*upper*	*time / gap %*	*nodes*	*time sep*	*nof ci*	*nof mci*	*nof fci*
di-yuan	none	3161	7391	7767	4.84	473766				
	ci + mci	3161	7621	7621	06:27	18707	00:06	259	0	
	all	3161	7621	7621	06:48	20017	00:04	252	0	6
nobel-eu	none	8588	8890	11192	20.6	1179254				
	ci + mci	8588	9062	14196	36.2	123530	32:49	507	2	
	all	8588	9063	12271	26.2	114033	43:22	455	2	140
nobel-germany	none	1474	1613	2115	23.7	2240597				
	ci + mci	1474	1796	1931	7.02	140206	31:50	271	1	
	all	1474	1802	2026	11.1	125403	39:11	272	1	70
sun	none	745.4	832.4	1038	19.8	63030				
	ci + mci	745.4	935.1	1067	12.4	30080	00:17	241	1	
	all	745.4	934.8	1066	12.3	29386	00:17	236	1	117
ta2	none	1064	1380	4605	70.0	12680				
	ci + mci	1064	1474	4188	64.8	5521	00:11	605	18	
	all	1064	1474	4188	64.8	5518	00:11	605	18	0
zib16	none	2097	2934	3315	11.5	343400				
	ci + mci	2097	2987	3314	9.88	84516	01:15	417	0	
	all	2097	2997	3261	8.11	70766	00:24	314	0	148
zib54	none	3813	11205	16333	31.4	9314				
	ci + mci	3813	13878	14830	6.42	5134	00:03	225	16	
	all	3813	13875	14903	6.90	5112	00:03	224	17	38

Table A.4: Results DIrected explicit link capacities

problem	*sep*	*lp*	*lower*	*upper*	*time/ gap*	*nodes*	*time sep*	*nof ci*	*nof mci*	*nof fci*
di-yuan	none	2746	5537	5537	02:02	18653				
	ci + mci	2746	5537	5537	04:25	19088	00:02	182	32	
	all	2746	5537	5537	02:59	12127	00:02	177	32	5
nobel-eu	none	6019	6149	6739	8.75	1098305				
	ci + mci	6019	6301	7738	18.6	184018	29:35	540	18	
	all	6019	6321	6803	7.08	146661	38:36	398	13	131
nobel-germany	none	1154	1226	1346	8.91	2970487				
	ci + mci	1154	1325	1325	17:38	140365	13:28	197	13	
	all	1154	1325	1325	12:02	79266	09:06	192	10	63
sun	none	571.6	614.3	699.9	12.22	80106				
	ci + mci	571.6	663.6	693.0	4.24	12119	00:04	273	32	
	all	571.6	663.4	693.0	4.26	11771	00:04	269	33	74
ta2	none	6724	9745	21375	54.4	31277				
	ci + mci	6724	9204	17675	47.9	678	00:02	734	152	
	all	6724	9204	17675	47.9	678	00:02	734	152	0
zib16	none	1582	2182	2182	09:44	60630				
	ci + mci	1582	2182	2182	10:25	23034	00:07	167	16	
	all	1582	2182	2182	10:54	19793	00:07	164	17	215
zib54	none	2018	7557	9950	24.0	16085				
	ci + mci	2018	8563	9994	14.3	5340	00:03	1116	469	
	all	2018	8563	9994	14.3	5340	00:03	1116	469	0

Table A.5: Results BIdirected explicit link capacities

problem	*sep*	*lp*	*lower*	*upper*	*time/ gap*	*nodes*	*time sep*	*nof ci*	*nof mci*	*nof fci*
di-yuan	none	3161	6566	6566	12:24	247328				
	ci + mci	3161	6566	6566	02:46	17471	00:00	163	10	
	all	3161	6566	6566	02:44	17471	00:00	163	10	0
nobel-eu	none	8588	8749	9532	8.214	1884632				
	ci + mci	8588	8968	9349	4.083	158449	32:47	577	17	
	all	8588	8962	9160	2.158	163275	32:22	515	14	25
nobel-germany	none	1474	1550	1674	7.365	2881728				
	ci + mci	1474	1619	1619	00:25	2976	00:24	93	14	
	all	1474	1619	1619	01:25	8212	01:19	159	19	4
sun	none	749.6	779.7	880.1	11.411	176617				
	ci + mci	749.6	858.7	863.7	0.579	27470	00:09	142	9	
	all	749.6	859.5	863.7	0.485	24395	00:08	141	8	115
ta2	none	1075	1232	2127	42.0	36103				
	ci + mci	1075	1310	1938	32.4	2662	00:07	1347	295	
	all	1075	1310	1938	32.4	2577	00:07	1325	287	0
zib16	none	2097	2708	2805	3.453	719285				
	ci + mci	2097	2800	2800	21:03	58292	00:09	322	20	
	all	2097	2800	2800	19:28	52523	00:05	273	19	52
zib54	none	3875	7671	10422	26.4	38733				
	ci + mci	3875	9383	10422	9.972	5952	00:03	1055	404	
	all	3875	9382	10422	9.980	5868	00:03	1053	403	0

Table A.6: Results UNdirected explicit link capacities

Appendix B

Proofs

The following definition and lemma is crucial for the understanding of the proofs of Theorem 4.23, 4.25 and 4.29.

Definition and Lemma B.1. Consider the cut set CS^{BI} in the single-facility case and suppose a feasible point p_0 is given such that all demand is routed

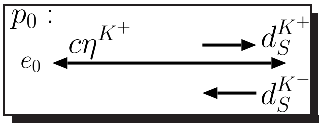

Figure B.1: All flow is routed on e_0.

on $e_0 = uv$ with capacity exactly $c\eta^{K^+}$, more precisely all flow for positive commodities is routed on uv and all flow for negative commodities is routed on vu (see Figure B.1). Assume that $d_S^{K^+} \geq |d_S^{K^-}|$ and $r^{K^+} < c$, which is equivalent to $d_S^{K^+} < c\eta^{K^+}$.

Hence the capacity on e_0 is not saturated. Let $|E_S| \geq 2$. To construct a second feasible point p, one unit of capacity is deleted on e_0. Note that $c(\eta^{K^+} - 1) = d_S^{K^+} - r^{K^+}$ (see Lemma 3.11).

1. If $d_S^{K^+} - r^{K^+} > 0$ there is still capacity on e_0. We decrease the flow on uv (with respect to K^+) by exactly r^{K^+} by changing flow for every positive commodity. Hence the capacity on uv is saturated. The flow on vu (with respect to K^-) is decreased by δ^{K^-}, where $\delta^{K^-} := |d_S^{K^-}| - (d_S^{K^+} - r^{K^+})$ if $|d_S^{K^-}| - (d_S^{K^+} - r^{K^+}) > 0$ and $0 < \delta^{K^-} < \min(|d_S^{K^-}|, r^{K^+})$ else.

   To do so we change the flow of every negative commodity. Note that $\delta^{K^-} \leq r^{K^+}$ since $d_S^{K^+} \geq |d_S^{K^-}|$.

   The missing flow is now routed on a second edge $e = ij$ with one unit of capacity (see Figure B.1).

2. If $d_S^{K^+} = r^{K^+} \iff \eta^{K^+} - 1 = 0$ we just copy the flow from e_0 to e. Set $\delta^{K^-} := \sum_{k \in K^-} d_S^k$ in this case.

This way we ensure that the new point is feasible, that the capacity on e is not saturated in both directions, that flows are positive on ij for $k \in K^+$ and that flows are positive on ji for $k \in K^-$.

In both cases we will denote by $\boldsymbol{\varphi_r^k}$ the amount of flow that has been rerouted for $k \in K$. It follows that $0 < \varphi_r^k \leq d_S^k$ for all $k \in K$, $\sum_{k \in K^+} \varphi_r^k = r^{K^+}$ and $\sum_{k \in K^-} \varphi_r^k = \delta^{K^-}$.

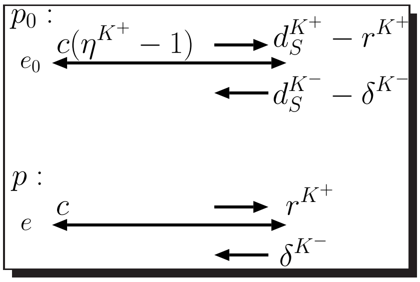

Figure B.2: e is used to reroute the flow.

If moreover $d_S^{K^+} - r^{K^+} > 0$, then $\varphi_r^k < d_S^k$ for all $k \in K$, flows are positive on uv for $k \in K^+$ and flows are positive on vu for $k \in K^-$.

There is another way to construct a feasible point p from p_0 if BIdirected capacity constraints are given. We delete flow and capacity the same way as above, but we construct a vector $\boldsymbol{\varphi_c}$ such that the total flow on ij is $\sum_{k \in K^+} \varphi_c^k = c$. On ji we route $\sum_{k \in K^+}(\varphi_c^k - \varphi_r^k) + \sum_{k \in K^-} \varphi_r^k$ such that the capacity on ji is not saturated if $d_S^{K^+} > |d_S^{K^-}|$ (see Figure B.1).

Notice that $\delta^{K^-} = r^{K^+}$ if $|d_S^{K^-}| = d_S^{K^+}$.

These constructions are used in the proofs of Theorem 4.25 and Theorem 4.29. In the proof of the following theorem (Theorem 4.23) a subset Q^+ of the positive commodities is considered initially routed on e_0. All the flow for $K^+ \backslash Q^+$ and $K^- \cup K^0$ is routed on a second edge and is not touched by the construction of points as above, the vectors φ_c and φ_r are defined with respect to $d_S^{Q^+}$ and r^{Q^+} then and there is only flow on uv and ij.

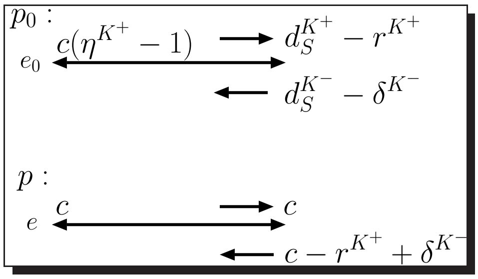

Figure B.3: e is used to reroute the flow in a different way.

B.1 Proof of Theorem 4.23

Proof. We will show that the related face

$$F_{BI} = \{\, (f,x) \in CS^{BI} : \ (f,x) \text{ satisfies (4.18) with equality} \,\}$$

is nontrivial and then by contradiction, we will show that it defines a facet (approach 2 for facet proofs Wolsey [1998, chap 9.2.3]).

Given $e = ij \in E_S$, let b_e denote the unit vector in $\mathbb{R}^{|E_S|+2|K||E_S|}$ for the integer design variable of e and let g_{ij}^k, g_{ji}^k be the unit vectors for the continuous flow variables of ij, ji for commodity $k \in K$.

We set $d := d_S^{Q^+}$, $\eta := \eta^{Q^+}$, $r := r^{Q^+} < c$ and $\epsilon > 0$ small enough. Let $\bar{Q}^+ := K^+ \backslash Q^+$. Choose $e_0 = uv \in E_1 \backslash E_2$ and $\bar{e}_0 = \bar{u}\bar{v} \in \bar{E}_1 \backslash E_2$. We construct a point on the face F_{BI} by sending all flow for Q^+ on e_0 and the flow for all other commodities on $\bar{e}_0$:

$$u_{e_0} := \eta b_{e_0} + \sum_{k \in Q^+} d^k g_{uv}^k + M b_{\bar{e}_0} + \sum_{k \in \bar{Q}^+} d^k g_{\bar{u}\bar{v}}^k + \sum_{k \in K^-} d^k g_{\bar{v}\bar{u}}^k.$$

where $M \in \mathbb{Z}_+$ is large enough. u_{e_0} is a feasible point of CS^{BI} since all demands are satisfied and the flow does not exceed the installed capacity. It is on the face F_{BI} because $x(E_1) = \eta$ and $x(E_2) = f^{Q^+}(\bar{E}_1^+) = f^{Q^+}(E_2^-) = 0$. Hence F_{BI} is not empty. $u_{e_0} + b_{e_0}$ is a point that is in CS^{BI} but not on the face F_{BI}.

We have shown that $\emptyset \neq F_{BI} \neq CS^{BI}$. It remains to show that F_{BI} is inclusionwise maximal. We do this by contradiction. Suppose there is face F of CS^{BI} with $F_{BI} \subset F$ and let F be defined by

$$\sum_{e=ij \in E_S} \beta_e x_e + \sum_{\substack{e=ij \in E_S \\ k \in K}} \gamma_{ij}^k f_{ij} + \sum_{\substack{e=ij \in E_S \\ k \in K}} \gamma_{ji}^k f_{ji} = \pi \tag{B.1}$$

where $\beta_e, \gamma^k_{ij}, \gamma^k_{ji}, \pi \in \mathbb{R}$. We will show that (B.1) is (4.18) up to a scalar multiple and a linear combination of flow conservation constraints, contradicting to $F_{BI} \subset F$.

Adding multiples of the $|K|$ flow conservation constraints to (B.1) gives $\gamma^k_{uv} = 0$ for all $k \in Q^+$, $\gamma^k_{\bar{u}\bar{v}} = 0$ for all $k \in \bar{Q}^+$ and $\gamma^k_{\bar{v}\bar{u}} = 0$ for all $k \in K^- \cup K^0$.

Set $\beta := \beta_{e_0}$ and $\bar{\beta} := \beta_{\bar{e}_0}$. Since u_{e_0} lies on the hyperplane, we conclude that $\beta\eta + \bar{\beta}M = \pi$. Now we modify u_{e_0} by installing a capacity of $M+1$ on $\bar{e}_0$. This is another point on the face and thus $\bar{\beta} = 0$. It follows that

$$\beta\eta = \pi \tag{B.1a}$$

The capacity on e_0 is not saturated since $d^{Q^+}_S < c\eta$. Modifying u_{e_0} by simultaneously increasing flow on uv and vu by ϵ for $k \in Q^+$ gives new points on the face and thus

$$\gamma^k_{uv}, \gamma^k_{vu} = 0 \qquad \forall k \in Q^+.$$

The same can be done on $\bar{u}\bar{v}$, $\bar{v}\bar{u}$ for $k \in K^- \cup K^0 \cup \bar{Q}^+$, hence

$$\gamma^k_{\bar{u}\bar{v}}, \gamma^k_{\bar{v}\bar{u}} = 0 \qquad \forall k \in K^- \cup K^0 \cup \bar{Q}^+$$

Now consider the disjoint partition $E_S := (E_1 \cap E_2) \cup (E_1 \backslash E_2) \cup (\bar{E}_1 \cap E_2) \cup (\bar{E}_1 \backslash E_2)$. We compute the coefficients $\gamma_e, \beta_{ij}, \beta_{ji}$ for $e = ij$ in each of the four sets by constructing new points. They obviously fulfil the flow conservation constraint and satisfy inequality (4.18) with equality. To see that they meet the BIdirected capacity constraints (4.11) just use that $r < c$ and remember the equations

$$c\eta = d + c - r \quad \text{and} \quad c(\eta - 1) = d - r \quad \text{(Lemma 3.11)}$$

Note that all the points defined with edges in $\bar{E}_1 \cap E_2$, $E_1 \backslash E_2$ and $\bar{E}_1 \backslash E_2$ additionally satisfy the UNdirected capacity constraints (4.12) Hence with $E_1 \cap E_2 = \emptyset$ the theorem holds for CS^{UN}.

$\boldsymbol{\bar{E}_1 \cap E_2 \neq \emptyset}$: For $e = ij \in \bar{E}_1 \cap E_2$ and $k \in Q^+$ define:

$$u_{e_0} + b_e + (c-r)g^k_{uv} + (c-r)g^k_{ji} \implies \beta\eta + \beta_e + (c-r)\gamma^k_{ji} = \pi \tag{B.1b}$$

$$u_{e_0} + b_{e_0} + cg^k_{uv} + b_e + cg^k_{ji} \implies \beta\eta + \beta + \beta_e + c\gamma^k_{ji} = \pi \tag{B.1c}$$

$$u_{e_0} + (c-r)g^k_{uv} + b_e + \tfrac{r}{2}g^k_{ij} + (c - \tfrac{r}{2})g^k_{ji} \implies \beta\eta + \beta_e + \tfrac{r}{2}\gamma^k_{ij} + (c - \tfrac{r}{2})\gamma^k_{ji} = \pi \tag{B.1d}$$

Comparison of (B.1b) and (B.1c) shows that $-r\gamma^k_{ji} = \beta$ for all $e \in \bar{E}_1 \cap E_2$, for all $k \in Q^+$. From (B.1b) it follows similarly that $\beta_e = \frac{\beta}{r}(c-r) \;\; \forall e \in \bar{E}_1 \cap E_2$. From (B.1d) we find that $\frac{\beta}{r}(c-r) - (c - \frac{r}{2})\frac{\beta}{r} + \frac{r}{2}\gamma^k_{ij} = 0$, which implies that $r\gamma^k_{ij} = \beta$ for all $e \in \bar{E}_1 \cap E_2$, for all $k \in Q^+$.

To conclude that $\gamma^k_{ji} = 0$ for all $k \in K^- \cup K^0$ just modify the point in (B.1b) by increasing flow for $k \in K^- \cup K^0$ on $\bar{u}\bar{v}$ by some ϵ and routing this ϵ-flow back on ji. Simultaneously increasing flow on ij, ji gives $\gamma^k_{ij} = 0$ for all $k \in K^- \cup K^0$. Since the same can be done for $k \in \bar{Q}^+$ on the arcs $\bar{u}\bar{v}, ij, ji$, we get $\gamma^k_{ij} = \gamma^k_{ji} = 0$ for all $k \in \bar{Q}^+$.

$\boldsymbol{E_1 \cap E_2 \neq \emptyset}$: For $e = ij \in E_1 \cap E_2$ and $k \in Q^+$ define:

$$v^k_e := u_{e_0} + b_e + cg^k_{ij} + cg^k_{ji} \implies \beta\eta + \beta_e + c\gamma^k_{ij} + c\gamma^k_{ji} = \pi \tag{B.1e}$$

We can still increase flow on uv by a small amount for every commodity in Q^+ . Decreasing flow on ij at the same time gives another point on the face and thus $\gamma^k_{ij} = 0$ for all $k \in Q^+$. When having changed v_e^k this way, some flow for $k \in K^- \cup K^0 \cup \bar{Q}^+$ can be routed on ij while the same amount of flow increases on $\bar{v}\bar{u}$. Hence $\gamma_{ij}^k = 0$ for all $k \in K^- \cup K^0 \cup \bar{Q}^+$.

For $k_1, k_2 \in Q^+, e = ij \in E_1 \cap E_2$ consider the point

$$v_e^{k_1} - \epsilon g_{uv}^{k_1} + \epsilon g_{uv}^{k_2} - \epsilon g_{ji}^{k_1} + \epsilon g_{ji}^{k_2}$$

It is well defined and feasible because flow on uv is positive for every $k \in Q^+$ and flow on ij is positive for k_1. It follows that $\gamma_{ji}^+ := \gamma_{ji}^{k_1} = \gamma_{ji}^{k_2}$ for all $k_1, k_2 \in Q^+$.

For the construction of the following vector see Definition and Lemma B.1. We modify u_{e_0} by deleting one unit of capacity for e_0 and rerouting flow on $e \in E_1 \cap E_2$:

$$\begin{aligned} & u_{e_0} - b_{e_0} - \sum_{k \in Q^+} \varphi_r^k g_{uv}^k + b_e + \sum_{k \in Q^+} \varphi_c^k g_{ij}^k + \sum_{k \in Q^+} (\varphi_c^k - \varphi_r^k) g_{ji}^k \\ \implies & \beta\eta - \beta + \beta_e + (c - r)\gamma_{ji}^+ = \pi \end{aligned} \tag{B.1f}$$

Note that $\sum_{k \in Q^+} (\varphi_c^k - \varphi_r^k) = c - r$ and $\gamma_{ij}^k = 0$. Comparing (B.1e) and (B.1f) gives

$$-r\gamma_{ji}^+ = -r\gamma_{ji}^k = \beta \ \forall k \in Q^+.$$

From (B.1e) and (B.1a) follows then

$$\beta_e = c\tfrac{\beta}{r} \ \forall e \in E_1 \cap E_2.$$

The total flow on ji in (B.1f) is $c - r$, thus the capacity on ji is not saturated. Increasing flow on $\bar{u}\bar{v}$ and ji for commodities in $\bar{Q}^+ \cup K^- \cup K^0$ gives $\gamma_{ji}^k = 0$ for all $k \in \bar{Q}^+ \cup K^- \cup K^0$.

$\boldsymbol{\bar{E}_1 \backslash E_2 \neq \emptyset}$: For $e = ij \in \bar{E}_1 \backslash E_2$ define:

$$u_{e_0} + b_e \quad \implies \quad \beta\eta + \beta_e = \pi \tag{B.1g}$$

The point can be modified by simultaneously increasing flow on uv and ji. This can be done for every commodity in Q^+, thus $\gamma_{ji}^k = 0$ for all $k \in Q^+$. Comparing (B.1g) with (B.1a) gives $\beta_e = 0$ for all $e \in \bar{E}_1 \backslash E_2$.

For the construction of the following vector see Definition and Lemma B.1. We modify u_{e_0} by deleting one unit of capacity for e_0 and rerouting flow on $e \in \bar{E}_1 \backslash E_2$:

$$\begin{aligned} & u_{e_0} - b_{e_0} - \sum_{k \in Q^+} \varphi_r^k g_{uv}^k + b_e + \sum_{k \in Q^+} \varphi_r^k g_{ij}^k \\ \implies & \beta\eta - \beta + \sum_{k \in Q^+} \varphi_r^k \gamma_{ij}^k = \pi \end{aligned} \tag{B.1h}$$

Modifying this point by simultaneously increasing flow on ij and $\bar{v}\bar{u}$ for $k \in K^- \cup K^0 \cup \bar{Q}^+$ gives $\gamma_{ij}^k = 0$ for all $k \in K^- \cup K^0 \cup \bar{Q}^+$. If $e \neq \bar{e}_0$, then simultaneously increasing flow on ij, ji gives $\gamma_{ji}^k = 0$ for all $k \in K^- \cup K^0 \cup \bar{Q}^+$.

It remains to show that $\gamma_{ij}^k = \frac{\beta}{r}$ for k in Q^+.

Assume first that $E_2 = \emptyset$. If $|Q^+| = 1$, it follows that $\beta\eta - \beta + r\gamma_{ij}^k = \pi$ and $r\gamma_{ij}^k = \beta$. If $|Q^+| > 1$ and $d_S^{K^+} > c$ flows are positive both on uv and ij for every commodity in Q^+ (see

Definition and Lemma B.1). We choose $k_1, k_2 \in Q^+$ and modify the point in (B.1h) by adding the flow $\epsilon g_{uv}^{k_2} - \epsilon g_{uv}^{k_1} + \epsilon g_{ij}^{k_1} - g_{ij}^{k_2}$. This way we conclude that $\gamma_{ij}^{k_1} = \gamma_{ij}^{k_2}$. From (B.1h) follows $r\gamma_{ij}^{k} = \beta$ for all $k \in Q^+$ since $\sum_{k \in Q^+} \varphi_r^k = r$.

Now let us assume that there is an edge $e = ij$ in $E_1 \cap E_2$. Modify the point in (B.1h) by installing one unit of capacity on $\bar{e}_0$ and sending a flow of c on ij and ji for a commodity $k_1 \in Q^+$. Now adding $\epsilon g_{ij}^{k_2} - \epsilon g_{ij}^{k_1} + \epsilon g_{ij}^{k_1} - g_{ij}^{k_2}$ gives $\gamma_{ij}^{k_1} = \gamma_{ij}^{k_2}$ and $r\gamma_{ij}^{k} = \beta$ for all $k \in Q^+$ again since $\gamma_{ij}^{k_1} = \gamma_{ij}^{k_2} = 0$.

Finally assume that there is $e = ij$ in $\bar{E}_1 \cap E_2$. For a commodity $k \in Q^+$ and $e = ij \in E_1 \cap E_2$ consider the following vector:

$$\begin{aligned} u_{e_0} + (c-r)g_{uv}^{k} + b_e + b_e + cg_{ji}^{k} + rg_{ij}^{k} \quad &\Longrightarrow \quad \beta\eta + \beta_e + \beta_e + c\gamma_{ji}^{k} + r\gamma_{ij}^{k} = \pi \\ &\Longrightarrow \quad \beta\eta + (c-r)\tfrac{\beta}{r} - c\tfrac{\beta}{r} + r\gamma_{ij}^{k} = \pi \\ &\Longrightarrow \quad \beta = r\gamma_{ij}^{k} \;\; \forall k \in Q^+ \end{aligned}$$

$\boldsymbol{E_1 \backslash E_2 \neq \emptyset}$: For $e = ij \in E_1 \backslash E_2$ we construct the following vector as in Definition and Lemma B.1:

$$\begin{aligned} u_{e_0} - b_{e_0} - \sum_{k \in K^+} \varphi_r^k g_{uv}^k + b_e + \sum_{k \in K^+} \varphi_r^k g_{ij}^k + \\ \Longrightarrow \beta\eta - \beta + \beta_e + \sum_{k \in K^+} \varphi_r^k \gamma_{ij}^k = \pi \end{aligned} \tag{B.1i}$$

For $k \in K$ add an ϵ-flow to ij and ji to conclude that $\gamma_{ij}^k = -\gamma_{ji}^k$ for all $k \in K$. If we can show that $\gamma_{ij}^k = 0$ for all $k \in K$ we are done because it follows that $\gamma_{ji}^k = 0$ for all $k \in K$ and $\beta_e = \beta$ for all $e \in E_1 \backslash E_2$.

For this modify the point (B.1i) by simultaneously increasing flow on ij and $\bar{v}\bar{u}$ for a k in K. Hence $\gamma_{ij}^k = -\gamma_{vu}^k = 0$.

Plugging in all coefficients we arrive at:

$$\begin{aligned} \beta x(E_1 \backslash E_2) \;&+\; \tfrac{\beta}{r} f(\bar{E}_1^+ \backslash E_2^+) \\ &+\; \tfrac{\beta}{r}(c-r)x(\bar{E}_1 \cap E_2) \;+\; \tfrac{\beta}{r} f(\bar{E}_1^+ \cap E_2^+) \;-\; \tfrac{\beta}{r} f(\bar{E}_1^- \cap E_2^-) \\ &+\; c\tfrac{\beta}{r} x(E_1 \cap E_2) \;-\; \tfrac{\beta}{r} f(E_1^- \cap E_2^-) \\ &=\; \beta\eta \end{aligned}$$

$\Longleftrightarrow$

$$f(\bar{E}_1^+) + cx(E_2) - f(E_2^-) + r\big(x(E_1) - x(E_2)\big) = r\eta$$

We have shown that the hyperplane (B.1) is a multiple of (4.18) plus a linear combination of flow conservation constraints. It follows that F_{BI} and F induce the same face. This is a contradiction. Hence F_{BI} is inclusionwise maximal and together with $\emptyset \neq F_{BI} \neq CS^{BI}$ it defines a facet of CS^{BI}. proper. This concludes the proof. ■

B.2 Proof of Theorem 4.25

Proof. We will show that the related face

$$F_{BI} = \{\, (f,x) \in CS^{BI} :\ (f,x) \text{ satisfies (4.18) at equality} \,\}$$

is nontrivial and then by contradiction, we will show that it defines a facets (approach 2 for facet proofs Wolsey [1998, chap 9.2.3]).

Given $e = ij \in E_S$, let b_e denote the unit vector in $\mathbb{R}^{|E_S|+2|K||E_S|}$ for the integer design variable of e and let g^k_{ij}, g^k_{ji} be the unit vectors for the continuous flow variables of ij, ji for commodity $k \in K$.

We set $d_S := d_S^{K^+}$, $\eta := \eta^{K^+}$, $r := r^{K^+} < c$ and $\epsilon > 0$ small enough. Choose $e_0 = uv \in E_1 \backslash E_2$ and consider a point as in Definition and Lemma B.1. All demand is satisfied by using only e_0 to send flow. In terms of the unit vectors this is:

$$u_{e_0} := \eta b_{e_0} + \sum_{k\in K^+} d^k_S g^k_{uv} + \sum_{k\in K^-} d^k_S g^k_{vu}.$$

u_{e_0} is a feasible point of CS^{BI} since all demands are satisfied and the flow does not exceed the installed capacity. It is on the face F_{BI} because $rx(E_1) = r\eta$. Hence F_{BI} is not empty. $u_{e_0} + b_{e_0}$ is a point that is in CS^{BI} but not on the face F_{BI}.

We have shown that $\emptyset \neq F_{BI} \neq CS^{BI}$. We still have to show that F_{BI} is inclusionwise maximal. We do this by contradiction. Suppose that there is a face F of CS^{BI} with $F_{BI} \subset F$ and let F be defined by the hyperplane

$$\sum_{e=ij\in E_S} \beta_e x_e + \sum_{\substack{e=ij\in E_S\\ k\in K}} \gamma^k_{ij} f_{ij} + \sum_{\substack{e=ij\in E_S\\ k\in K}} \gamma^k_{ji} f_{ji} = \pi \tag{B.2}$$

where $\beta_e, \gamma^k_{ij}, \gamma^k_{ji}, \pi \in \mathbb{R}$. We will show that (B.2) is completely described by (4.18) up to a scalar multiple and a linear combination of flow conservation constraints, contradicting $F_{BI} \subset F$.

Adding multiples of the $|K|$ flow conservation constraints to (B.2) gives $\gamma^k_{uv} = 0\ \forall k \in K^+$ and $\gamma^k_{vu} = 0\ \forall k \in K^- \cup K^0$ w. l. o. g..

Set $\beta := \beta_{e_0}$. Since u_{e_0} lies on the hyperplane, we conclude that

$$\beta\eta = \pi \tag{B.2a}$$

Modifying u_{e_0} by simultaneously increasing flow on uv and vu by ϵ for every commodity gives new points on the face and thus $\gamma^k_{uv}, \gamma^k_{vu} = 0\ \forall k \in K$.

Now consider the disjoint partition $E_S := (\bar{E}_1 \cap E_2) \cup (E_1 \cap E_2) \cup (E_1 \backslash E_2)$. (Note that $\bar{E}_1 \backslash E_2 = \emptyset$). We calculate the coefficients $\gamma_e, \beta_{ij}, \beta_{ji}$ for $e = ij$ in each of the three sets by constructing new points. The fact that all the points are on the face F_{BI} will not be mentioned anymore. They obviously fulfil the flow conservation constraint and satisfy inequality (4.18) with equality. To see that they meet the BIdirected capacity constraints (4.11) just use that $c(\eta+1) = d_S + 2c - r$, $c\eta = d_S + c - r$, $c(\eta - 1) = d_S - r$ (see Lemma 3.11) and $r < c$.

Note that all the points defined with edges in $\bar{E}_1 \cap E_2$ and $E_1 \backslash E_2$ additionally satisfy the UNdirected capacity constraints (4.12) when no negative commodities are given. Hence with $E_1 \cap E_2 = \emptyset$ and $K^- = \emptyset$ the theorem holds for CS^{UN}.

$\bar{\boldsymbol{E}}_1 \cap \boldsymbol{E_2} \neq \emptyset$: For $e = ij \in \bar{E}_1 \cap E_2$ and $k \in K^+$ define:

$$u_{e_0} + b_e + (c-r)g^k_{uv} + (c-r)g^k_{ji} \implies \beta\eta + \beta_e + (c-r)\gamma^k_{ji} = \pi \tag{B.2b}$$
$$u_{e_0} + b_{e_0} + cg^k_{uv} + b_e + cg^k_{ji} \implies \beta\eta + \beta + \beta_e + c\gamma^k_{ji} = \pi \tag{B.2c}$$
$$u_{e_0} + (c-r)g^k_{uv} + b_e + \tfrac{r}{2}g^k_{ij} + (c-\tfrac{r}{2})g^k_{ji} \implies \beta\eta + \beta_e + \tfrac{r}{2}\gamma^k_{ij} + (c-\tfrac{r}{2})\gamma^k_{ji} = \pi \tag{B.2d}$$

Comparing (B.2b) and (B.2c) shows that $-r\gamma^k_{ji} = \beta \;\; \forall e \in \bar{E}_1 \cap E_2, \;\; \forall k \in K^+$. From (B.2b) it follows similarly that $\beta_e = \frac{\beta}{r}(c-r) \;\; \forall e \in \bar{E}_1 \cap E_2$. From (B.2d) we find that $\frac{\beta}{r}(c-r) - (c-\frac{r}{2})\frac{\beta}{r} + \frac{r}{2}\gamma^k_{ij} = 0 \iff r\gamma^k_{ij} = \beta \;\; \forall e \in \bar{E}_1 \cap E_2, \;\; \forall k \in K^+$.

To conclude that $\gamma^k_{ij} = 0 \;\; \forall k \in K^- \cup K^0$ just modify the point in (B.2b) by increasing flow for $k \in K^- \cup K^0$ on vu and ij by some ϵ. Finally modify the point by simultaneously increasing flow on ij and ji by a small amount for every $k \in K^-$. This gives $\gamma^k_{ji} = 0 \;\; \forall e \in \bar{E}_1 \cap E_2, \;\; \forall k \in K^- \cup K^0$.

$\boldsymbol{E_1} \cap \boldsymbol{E_2} \neq \emptyset$: We can assume $K^0 = \emptyset$ here. For $e = ij \in E_1 \cap E_2$ and $k \in K^+$ define:

$$v^k_e := u_{e_0} + b_e + cg^k_{ij} + cg^k_{ji} \implies \beta\eta + \beta_e + c\gamma^k_{ij} + c\gamma^k_{ji} = \pi \tag{B.2e}$$

We can still increase flow on uv by a small amount for every commodity in K^+ . Decreasing flow on ij at the same time gives another point on the face and thus $\gamma^k_{ij} = 0 \;\; \forall k \in K^+$. When having changed v^k_e this way, some flow for $k \in K^-$ can be routed on ij while the same amount of flow increases on vu. Hence $\gamma^k_{ij} = 0 \;\; \forall k \in K^-$.

For $k_1, k_2 \in K^+, e = ij \in E_1 \cap E_2$ consider the point

$$v^{k_1}_e - \epsilon g^{k_1}_{uv} + \epsilon g^{k_2}_{uv} - \epsilon g^{k_1}_{ji} + \epsilon g^{k_2}_{ji}$$

It is well defined and feasible because flow on uv is positive for every $k \in K^+$ and flow on ij is positive for k_1. It follows that $\gamma^+_{ji} := \gamma^{k_1}_{ji} = \gamma^{k_2}_{ji} \;\; \forall k_1, k_2 \in K^+$.

For the construction of the following vector see Definition and Lemma B.1. We modify u_{e_0} by deleting one unit of capacity for e_0 and rerouting flow on $e \in E_1 \cap E_2$:

$$u_{e_0} - b_{e_0} - \sum_{k \in K^+} \varphi^k_r g^k_{uv} - \sum_{k \in K^-} \varphi^k_r g^k_{vu} + b_e + \sum_{k \in K^+} \varphi^k_c g^k_{ij} + \sum_{k \in K^+} (\varphi^k_c - \varphi^k_r) g^k_{ji} + \sum_{k \in K^-} \varphi^k_r g^k_{ji}$$
$$\implies \beta\eta - \beta + \beta_e + (c-r)\gamma^+_{ji} + \sum_{k \in K^-} \varphi^k_r \gamma^k_{ji} = \pi \tag{B.2f}$$

If $|K^-| = 0$, we compare (B.2e) and (B.2f) and conclude $-r\gamma^k_{ji} = \beta \;\; \forall k \in K^+$ From (B.2e) and (B.2a) follows then $\beta_e = c\frac{\beta}{r} \;\; \forall e \in E_1 \cap E_2$.

Else if $|K^-| > 0$, $d_S = d^{K^+}_S > |d^{K^-}_S|$ and $d^{K^+}_S > c$ there is still capacity on $s = uv$ and the vector φ^k_r can be constructed such that arc vu is not saturated (see Definition and Lemma B.1). We can increase flow on vu and decrease it on ji which gives $\gamma^k_{ji} = 0 \;\; \forall k \in K^-$ and thus $-r\gamma^k_{ji} = \beta \;\; \forall k \in K^+$ and $\beta_e = c\frac{\beta}{r} \;\; \forall e \in E_1 \cap E_2$ as above.

$\boldsymbol{E_1} \backslash \boldsymbol{E_2} \neq \emptyset$: For $e = ij \in E_1 \backslash E_2$ we construct the following vector as in Definition and Lemma B.1:

$$u_{e_0} - b_{e_0} - \sum_{k \in K^+} \varphi^k_r g^k_{uv} - \sum_{k \in K^-} \varphi^k_r g^k_{vu} + b_e + \sum_{k \in K^+} \varphi^k_r g^k_{ij} + \sum_{k \in K^-} \varphi^k_r g^k_{ji}$$
$$\implies \beta\eta - \beta + \beta_e + \sum_{k \in K^+} \varphi^k_r \gamma^k_{ij} + \sum_{k \in K^-} \varphi^k_r \gamma^k_{ji} = \pi \tag{B.2i}$$

For $k \in K$ add an ϵ-flow to ij and ji to conclude that $\gamma_{ij}^k = -\gamma_{ji}^k \ \forall k \in K$. If we can show that $\gamma_{ij}^k = 0 \vee \gamma_{ji}^k = 0 \ \forall k \in K$ we are done because it follows that $\beta_e = \beta \ \forall e \in E_1 \backslash E_2$.

All conditions imply that $E_2 \neq \emptyset$. First suppose that there is $\bar{e}_0 = \bar{u}\bar{v}$ in $E_1 \cap E_2$. Modify the point in (B.2i) by installing one unit of capacity on $\bar{e}_0$ and sending a flow of c on $\bar{u}\bar{v}$ and $\bar{v}\bar{u}$ for a commodity $k \in K^+$, which gives a point on the face. Now decrease flow on $\bar{u}\bar{v}$ and increase it on ij by ϵ. Hence $\gamma_{ij}^k = 0 \ \forall k \in K^+$ since $\gamma_{\bar{u}\bar{v}}^k = 0 \ \forall k \in K$. Having done so simultaneously increasing flow on $\bar{u}\bar{v}$ and ji gives $\gamma_{ji}^k = 0 \ \forall k \in K^- \cup K^0$.

Finally suppose that there is $\bar{e}_0 = \bar{u}\bar{v}$ in $\bar{E}_1 \cap E_2$. For $k \in K^+$ consider the vector

$$u_{e_0} + (c-r)g_{uv}^k + b_{\bar{u}\bar{v}} + b_e + cg_{\bar{v}\bar{u}}^k + rg_{ij}^k$$

Simultaneously increasing flow on vu and on ij gives $\gamma_{ij}^k = 0 \ \forall k \in K$.

Plugging in all coefficients we arrive at:

$$\begin{aligned}
\beta x(E_1 \backslash E_2) \ &+ \ \tfrac{\beta}{r} f(\bar{E}_1^+ \backslash E_2^+) \\
&+ \ \tfrac{\beta}{r}(c-r)x(\bar{E}_1 \cap E_2) \ + \ \tfrac{\beta}{r} f(\bar{E}_1^+ \cap E_2^+) \ - \ \tfrac{\beta}{r} f(\bar{E}_1^- \cap E_2^-) \\
&+ \ c\tfrac{\beta}{r} x(E_1 \cap E_2) \qquad\quad - \ \tfrac{\beta}{r} f(E_1^- \cap E_2^-) \\
&= \ \beta\eta
\end{aligned}$$

$\Longleftrightarrow$

$$f(\bar{E}_1^+) + cx(E_2) - f(E_2^-) + r\big(x(E_1) - x(E_2)\big) = r\eta$$

We have shown that the hyperplane (B.2) is a multiple of (4.18) plus a linear combination of flow conservation constraints. It follows that F_{BI} and F induce the same face, which is a contradiction. Hence F_{BI} is inclusionwise maximal and with $\emptyset \neq F_{BI} \neq CS^{BI}$ it defines a facet of CS^{BI}. This concludes the proof. ∎

B.3 Proof of Theorem 4.29

Proof. Sufficiency: If ($E_1 = \emptyset$ and $d_S^{K^+} < c$), then (4.21) reduces to the cut inequality (4.23) which is facet-defining for CS^{BI} if $|E_S| = 1$. The same happens when $\bar{E}_1 = \emptyset$, (4.21) reduces to the cut inequality (4.23), which is facet-defining if $|E_S| = 1$ or $d_S^{K^+} > c$ (see Theorem 4.21).

For the rest of the proof we can assume that $E_1, \bar{E}_1 \neq \emptyset$.

We use the same technique as in the proof of Theorem 4.23 (approach 2 for facet proofs Wolsey [1998, chap 9.2.3]).

We will show that the related face

$$F_{BI} = \{\, (f,x) \in CS^{BI} : \ (f,x) \text{ satisfies (4.21) with equality} \,\}$$

is nontrivial and then by contradiction, we will show that it defines a facets.

Given $e = ij \in E_S$ let b_e denote the unit vector in $\mathbb{R}^{|E_S|+2|K||E_S|}$ for the design variable of e and let g_{ij}^k, g_{ji}^k be the unit vectors for the flow variables for commodity $k \in K$ of e in both directions.

Suppose $r^{K^+} < c$ and $E_1, \bar{E}_1 \neq \emptyset$. We set $d_S := d_S^{K^+}$, $\eta := \eta^{K^+}$, $r := r^{K^+} < c$ and $\epsilon > 0$ small enough.

Choose $e_0 = uv \in E_1 \backslash E_2$ and $\bar{e}_0 = \bar{u}\bar{v} \in \bar{E}_1$. Consider a point as in Definition and Lemma B.1, all demand is satisfied by using only e_0 to send flow. In terms of the unit vectors this is:

$$u_{e_0} \ := \eta b_{e_0} + \sum_{k \in K^+} d_S^k g_{uv}^k + \sum_{k \in K^-} d_S^k g_{vu}^k.$$

u_{e_0} is a feasible point of CS^{BI} (see proof of Theorem 4.25). It is on the face F_{BI} because $cx(E_1) - d_S = c\eta - d_S = c - r$ (see Lemma 3.11). Hence F_{BI} is not empty. $u_{e_0} + b_{e_0}$ is a point that is in CS^{BI} but not on the face F_{BI}.

We have shown that $\emptyset \neq F_{BI} \neq CS^{BI}$. It remains to show that F_{BI} is inclusionwise maximal. We do this by contradiction. Suppose that there is a face F of CS^{BI} with $F_{BI} \subset F$ and let F be defined by the hyperplane

$$\sum_{e=ij \in E_S} \beta_e x_e + \sum_{\substack{e=ij \in E_S \\ k \in K}} \gamma_{ij}^k f_{ij} + \sum_{\substack{e=ij \in E_S \\ k \in K}} \gamma_{ji}^k f_{ji} = \pi \tag{B.3}$$

where $\beta_e, \gamma_{ij}^k, \gamma_{ji}^k, \pi \in \mathbb{R}$. We will show that (B.3) is (4.21) up to a scalar multiple and a linear combination of flow conservation constraints, contradicting $F_{BI} \subset F$.

We may add multiples of the $|K|$ flow conservation constraint to (B.3). Therefore we assume that $\gamma_{\bar{v}\bar{u}}^k = 0$ for all $k \in K$ w. l. o. g..

Set $\beta := \beta_{e_0}$ and $\bar{\beta} := \beta_{\bar{e}_0}$. Since u_{e_0} lies on the hyperplane, we conclude that

$$\beta\eta + \sum_{k \in K^+} d_S^k \gamma_{uv}^k + \sum_{k \in K^-} d_S^k \gamma_{vu}^k = \pi \tag{B.3a}$$

Modifying u_{e_0} by simultaneously increasing flow on uv and vu by ϵ for every commodity gives new points on the face ($c\eta - d_S + \epsilon - \epsilon = c - r$) and thus $\gamma_{uv}^k = -\gamma_{vu}^k$ for all $k \in K$.

Now consider the disjoint partition $E_S := E_1 \cup \bar{E}_1$. We calculate the coefficients $\gamma_e, \beta_{ij}, \beta_{ji}$ for $e = ij$ in each of the two sets by constructing new points. It will not always be mentioned that all the points are on the face F_{BI}. In most of the cases they obviously fulfil the flow conservation constraint and satisfy inequality (4.21) with equality. To see that they meet the BIdirected capacity constraints (4.11) just use that $c\eta = d_S + c - r$, $c(\eta - 1) = d_S - r$ (see Lemma 3.11) and $r < c$.

If $K^- = \emptyset$, then all the points additionally satisfy UNdirected capacity constraints. With $K^- = \emptyset$ the theorem holds for CS^{UN}.

First we define a vector u_e for all $e \in E_S$, see Definition and Lemma B.1 for this construction. We modify u_{e_0} by deleting one unit of capacity for e_0 and rerouting flow on $e \in E_S$:

$$u_e := u_{e_0} - b_{e_0} - \sum_{k \in K^+} \varphi_r^k g_{uv}^k - \sum_{k \in K^-} \varphi_r^k g_{vu}^k + b_e + \sum_{k \in K^+} \varphi_r^k g_{ij}^k + \sum_{k \in K^-} \varphi_r^k g_{ji}^k$$

$$\Longrightarrow$$

$$\beta\eta - \beta + \sum_{k \in K^+} (d_S^k - \varphi_r^k)\gamma_{uv}^k + \sum_{k \in K^-} (d_S^k - \varphi_r^k)\gamma_{vu}^k + \beta_e + \sum_{k \in K^+} \varphi_r^k \gamma_{ij}^k + \sum_{k \in K^-} \varphi_r^k \gamma_{ji}^k = \pi. \tag{B.3b}$$

Modifying u_e by simultaneously increasing flow on ij and ji by ϵ for every commodity gives

$$\gamma_{ij}^k = -\gamma_{ji}^k \ \ \forall e = ij \in E_S, k \in K.$$

$\boldsymbol{E_1} \neq \boldsymbol{\emptyset}$: For $e = ij \in E_1$ and $k^* \in K^+$ consider:

$$v_e := u_e + (c-r)g_{ij}^{k^*} + b_{\bar{e}_0} + (c-r)g_{\bar{v}\bar{u}}^{k^*}$$
$$\Longrightarrow$$
$$\beta\eta - \beta + \sum_{k\in K^+}(d_S^k - \varphi_r^k)\gamma_{uv}^k + \sum_{k\in K^-}(d_S^k - \varphi_r^k)\gamma_{vu}^k + \beta_e + \sum_{k\in K^+}\varphi_r^k\gamma_{ij}^k + \sum_{k\in K^-}\varphi_r^k\gamma_{ji}^k + \bar{\beta} + (c-r)\gamma_{ij}^{k^*} = \pi. \tag{B.3c}$$

Remember that $\gamma_{\bar{v}\bar{u}}^{k^*} = 0$ and note that in v_e the total flow on ij equals c, since $\sum_{k\in K^+}\varphi_r^k = r$ (see Definition and Lemma B.1). With $x(E_1) = \eta$, $x(\bar{E}_1) = 1$ and $f^{K^+}(E_1) = d_S - r + c$, the point is on the face. By comparing (B.3b) and (B.3c) we conclude that

$$\gamma_{ji}^k = -\gamma_{ij}^k = \frac{\bar{\beta}}{c-r} \;\; \forall e = ij \in E_1, k \in K^+.$$

For $k \in K^- \cup K^0$ we modify v_e by increasing flow on ji and $\bar{u}\bar{v}$. With $\gamma_{\bar{v}\bar{u}}^k = -\gamma_{\bar{u}\bar{v}}^k = 0$ as assumed, we get

$$\gamma_{ji}^k = -\gamma_{ij}^k = 0 \;\; \forall e = ij \in E_1, k \in K^- \cup K^0.$$

(B.3a) and (B.3b) (with $e \in E_1$) now reduce to

$$\beta\eta - d_S\frac{\bar{\beta}}{c-r} = \pi \quad \text{and} \quad \beta\eta - \beta - d_S\frac{\bar{\beta}}{c-r} + \beta_e = \pi \tag{B.3d}$$

It follows

$$\beta_e = \beta \;\; \forall e \in E_1.$$

$\boldsymbol{\bar{E}_1} \neq \boldsymbol{\emptyset}$: For $e = ij \in \bar{E}_1$ and $k^* \in K^+$ define:

$$w_e := u_{e_0} + (c-r)g_{uv}^{k^*} + b_e + (c-r)g_{ji}^{k^*}$$
$$\Longrightarrow$$
$$\beta\eta + \sum_{k\in K^+} d_S^k\gamma_{uv}^k + (c-r)\gamma_{uv}^{k^*} + \beta_e + (c-r)\gamma_{ji}^{k^*} = \pi. \tag{B.3e}$$

For $k \in K^- \cup K^0$ increasing flow on vu and ij gives

$$\gamma_{ji}^k = -\gamma_{ij}^k = \gamma_{vu}^k = 0 \;\; \forall e = ij \in \bar{E}_1, k \in K^- \cup K^0.$$

For $k \in K^+$ we modify w_e by simultaneously increasing flow on uv, ji by ϵ and at the same time decreasing flow for k^* on uv, ji by the same amount. Hence

$$-\gamma_{ij}^k = \gamma_{ji}^k = \gamma_{ji}^{k^*} = -\gamma_{ij}^{k^*} \;\; \forall e = ij \in \bar{E}_1, k \in K.$$

(B.3b) with $e \in \bar{E}_1$ now reduces to

$$\beta\eta - \beta - (d_S - r)\frac{\bar{\beta}}{c-r} + \beta_e + r\gamma_{ij}^{k^*} = \pi. \tag{B.3f}$$

(B.3e) can be written as

$$\beta\eta - (d_S + c - r)\frac{\bar{\beta}}{c-r} + \beta_e - (c-r)\gamma_{ij}^{k^*} = \pi. \tag{B.3g}$$

Evaluating (B.3f) for $e = \bar{e}_0 = \bar{u}\bar{v}$ and comparing with (B.3a) gives

$$\beta = \frac{c\bar{\beta}}{c-r}$$

since $\beta_{\bar{e}_0} = \bar{\beta}$ and $\gamma_{\bar{u}\bar{v}}^k = 0$ for all $k \in K$. Then from (B.3f) and (B.3g) follows that $r\gamma_{ij}^{k^*} = (r-c)\gamma_{ij}^{k^*}$ But $c > r > 0$ and thus

$$\gamma_{ji}^k = -\gamma_{ij}^k = 0 \;\; \forall e = ij \in \bar{E}_1, k \in K.$$

Now comparing (B.3g) with (B.3a) results in

$$\beta_e = \bar{\beta} \;\; \forall e \in \bar{E}_1.$$

Plugging in all coefficients we arrive at:

$$\frac{c\bar{\beta}}{c-r}x(E_1) - \frac{\bar{\beta}}{c-r}f(E_1^+) + \frac{\bar{\beta}}{c-r}f(E_1^-) + \bar{\beta}x(\bar{E}_1) = \bar{\beta}$$

$\iff$

$$cx(E_1) - f(E_1^+) + f(E_1^-) + (c-r)x(\bar{E}_1) = (c-r).$$

We have shown that the hyperplane (B.3) is a multiple of (4.21) plus a linear combination of flow conservation constraints. It follows that F_{BI} and F induce the same face, which is a contradiction. Hence F_{BI} is inclusionwise maximal and together with $\emptyset \neq F_{BI} \neq CS^{BI}$ it defines a facet of CS^{BI}. ■

Appendix C

Notation

$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$	real, rational, integer numbers
$\mathbb{R}_+$, $\mathbb{Q}_+$, $\mathbb{Z}_+$	nonnegative real, rational, integer numbers
V	nodes
A, E	arcs, edges
a, e	arc, edge
(i,j), ij	arc with source i and target j, edge with endnodes i and j
$G=(V,A)$	directed graph
$G=(V,E)$	undirected graph
S	node set
CS^{DI}, CS^{BI}, CS^{UN}	DIrected, BIdirected, UNdirected cut set polyhedron
$G[S]$	subgraph induced by S
$A[S]$, $E[S]$	arcs, edges with both endnodes in S
A_S, E_S	directed, undirected cut defined by S
A_S^+, A_S^-	arcs from S to $V\backslash S$, arcs from $V\backslash S$ to S
A_1^+, A_2^-	subset of A_S^+, subset of A_S^-
E_1, E_2	subsets of E_S
K, Q	set of all commodities, subset of all commodities
K^+, K^-, K^0	positive, negative, zero commodities w. r. t. a node set S
T	set of technologies (facilities, link designs)
f	continuous variable – often denotes flow
x	integer variable – often denotes the number of installed link designs
c, d, κ	capacity, demand, cost
$\langle a \rangle$	$a - \lfloor a \rfloor$
$\lceil a \rceil$	smallest integer greater than or equal to a
$\lfloor a \rfloor$	greatest integer smaller than or equal to a
$r(d,c)$	$d - c(\lceil \frac{d}{c} \rceil - 1)$
$\mathcal{G}_{d,c}(a)$	$r(d,c)\lceil \frac{a}{c} \rceil - (r(d,c) - r(a,c))^+$ — subadditive *MIR*-function
$\mathcal{F}_{d,c}(a)$	$r(-d,c)\lfloor \frac{a}{c} \rfloor + (r(-d,c) - r(-a,c))^+$ — superadditive *MIR*-function
M, N, L, R, U, C	often denote index sets
(C^+, C^-), (P^+, P^-)	flow cover, flow pack

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