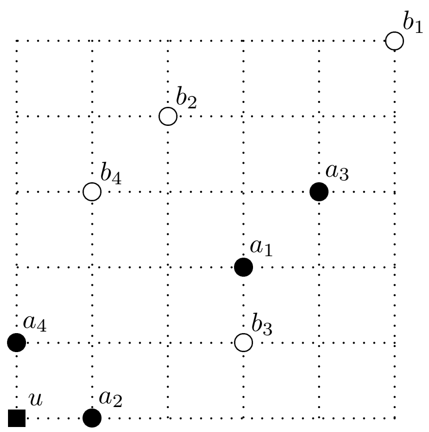
Jörg Rambau / Luis M. Torres

Homework

1. In the following street network, four transportation requests from a_i to b_i (where $i \in \{1, \dots 4\}$) are given. The dotted lines represent the streets, and the distance between two adjacent crossings is always 1. Consider the problem of attending these requests with one unit u located initially at the lowest left corner of the map. We want to find the shortest feasible service tour for u, starting and ending at its current position. The unit can transport only one request at the same time. Moreover, we do not allow *preemption*, i.e., once a request has been picked up, it can be only dropped at its destination.



   (a) Model this problem as an asymmetric traveling salesman problem (ATSP).
   (b) Give an IP formulation of the problem.
   (c) Use branch-and-bound to solve the IP to optimality. Draw the search tree indicating the branching variables and the LP lower bounds.

(3 pts)

2. Let $D = (V, A)$ be a directed graph and $c : A \to \mathbb{R}_+$ a nonnegative cost function on the arcs.

   (a) Suppose a set of labels $\{y_v : v \in V\}$ has been found for all the vertices in V such that
   $$y_v + c_{vw} \geq y_w$$
   holds for all $vw \in A$. Let s and t be two arbitrary nodes in V. Prove that any path from s to t has a cost larger or equal than $y_t - y_s$.

(b) Prove that the distance labels found by Djikstra's algorithm satisfy this inequality for all arcs in A.

(c) Use the last observation to prove the correctness of Djikstra's algortihm and show that it can be implemented to run in time $O(n^2)$.

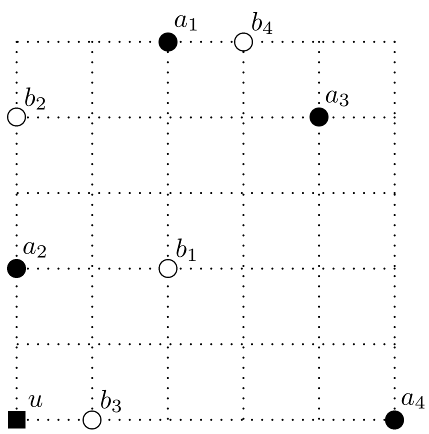
(4 pts)

3. Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$. Suppose that $S \subseteq A$, $T \subseteq B$, and there is a matching M_1 covering S and a matching M_2 covering T. Prove that then there is a matching covering $S \cup T$.

(3 pts)

Homework

1. In the following street network, four transportation requests from a_i to b_i (where $i \in \{1, \ldots 4\}$) are given. The dotted lines represent the streets, and the distance between two adjacent crossings is always 1. Consider the problem of attending these requests with one unit u located initially at the lowest left corner of the map. We want to find the shortest feasible service tour for u, starting and ending at its current position. The unit can transport only one request at the same time. Moreover, we do not allow *preemption*, i.e., once a request has been picked up, it can be only dropped at its destination.



   (a) Model this problem as an asymmetric traveling salesman problem (ATSP).

   (b) Give an IP formulation of the problem.

   (c) Use branch-and-bound to solve the IP to optimality. Draw the search tree indicating the branching variables and the LP lower bounds.

(3 pts)

2. Prove the correctness of Kruskal's algorithm to find the minimum-weight spanning tree in a graph.

(4 pts)

3. Let E be a finite set and $\mathcal{B} \subseteq 2^E$ a family of subsets of E. Show that $\mathcal{B}$ is the set of bases of some matroid $(E, \mathcal{F})$ if and only if the following holds:

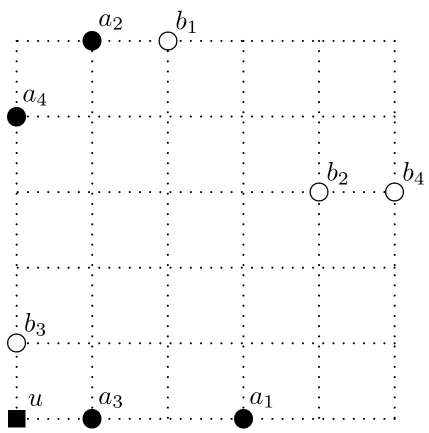
(B1) $\mathcal{B} \neq \emptyset$

(B2) For any $B_1, B_2 \in \mathcal{B}$ and $y \in B_2 \setminus B_1$ there exists an $x \in B_1 \setminus B_2$ with $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$

(3 pts)

Homework

1. In the following street network, four transportation requests from a_i to b_i (where $i \in \{1, \ldots 4\}$) are given. The dotted lines represent the streets, and the distance between two adjacent crossings is always 1. Consider the problem of attending these requests with one unit u located initially at the lowest left corner of the map. We want to find the shortest feasible service tour for u, starting and ending at its current position. The unit can transport only one request at the same time. Moreover, we do not allow *preemption*, i.e., once a request has been picked up, it can be only dropped at its destination.



   (a) Model this problem as an asymmetric traveling salesman problem (ATSP).

   (b) Give an IP formulation of the problem.

   (c) Use branch-and-bound to solve the IP to optimality. Draw the search tree indicating the branching variables and the LP lower bounds.

(3 pts)

2. A *topological sort* of a digraph $D = (V, A)$ is a set $\{\ell_v : v \in V\}$ of labels for its nodes having the property that $\ell_v < \ell_w$ holds for all arcs $vw \in A$.

   (a) Show that D has a topological sort if an only if it doesn't contain any (directed) cycle.

(b) Give an $O(m)$ algorithm for finding a topological sort in an acyclic digraph and analyze its complexity. Consider for this purpose that the operation of deleting a node from a graph can be implemented to run in constant time.

(4 pts)

3. Let G be a graph and P the *fractional perfect matching polytope* of G, defined by

$$P := \left\{ x \in [0,1]^E : \sum_{e \in \delta(v)} x_e = 1 \forall v \in V \right\}.$$

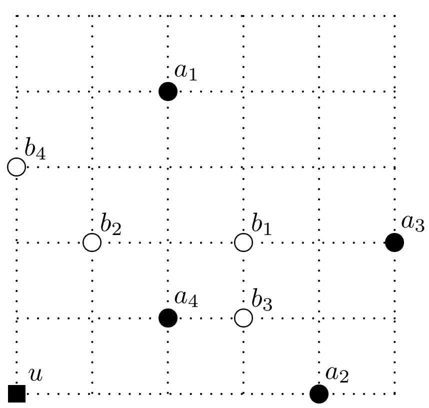
Prove that P is *half-integral*, i.e., that the coordinates of its vertices may only take values of $0, 1$ and $\frac{1}{2}$. Moreover, show that the edges corresponding to fractional coordinates form a set of disjoint odd circuits.

Hint: Recall that given any vertex x of a polytope $P = (A, b)$, the rows of A corresponding to the inequalities satisfied by x with equality form a submatrix of full rank.

(3 pts)

Homework

1. In the following street network, four transportation requests from a_i to b_i (where $i \in \{1, \dots 4\}$) are given. The dotted lines represent the streets, and the distance between two adjacent crossings is always 1. Consider the problem of attending these requests with one unit u located initially at the lowest left corner of the map. We want to find the shortest feasible service tour for u, starting and ending at its current position. The unit can transport only one request at the same time. Moreover, we do not allow *preemption*, i.e., once a request has been picked up, it can be only dropped at its destination.



   (a) Model this problem as an asymmetric traveling salesman problem (ATSP).
   (b) Give an IP formulation of the problem.
   (c) Use branch-and-bound to solve the IP to optimality. Draw the search tree indicating the branching variables and the LP lower bounds.

(3 pts)

2. Let $G = (V, E)$ be an undirected graph and $A \subset V$ a set of nodes. A connected component of the graph $G \setminus A$ which has an odd number of vertices will be called an *odd component* of $G \setminus A$. Moreover, let us denote by $\operatorname{oc}(G \setminus A)$ the number of such odd components.

   Prove that, for any matching M in G, the following holds:

$$|M| \leq \frac{1}{2}\left(|V| - \mathrm{oc}\,(G \setminus A) + |A|\right).$$

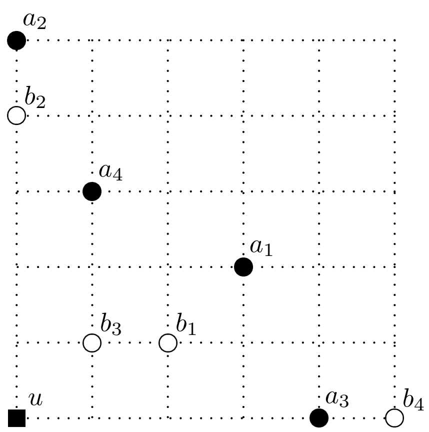
(4 pts)

3. Prove *Carathéodory's theorem*: If $X \subseteq \mathbb{R}^n$ and $y \in \mathrm{conv}(X)$, then there are $x_1, \ldots x_n \in X$ such that $y \in \mathrm{conv}(\{x_1, \ldots, x_n\})$.

(3 pts)

Homework

1. In the following street network, four transportation requests from a_i to b_i (where $i \in \{1, \ldots 4\}$) are given. The dotted lines represent the streets, and the distance between two adjacent crossings is always 1. Consider the problem of attending these requests with one unit u located initially at the lowest left corner of the map. We want to find the shortest feasible service tour for u, starting and ending at its current position. The unit can transport only one request at the same time. Moreover, we do not allow *preemption*, i.e., once a request has been picked up, it can be only dropped at its destination.



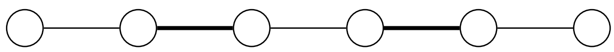
   (a) Model this problem as an asymmetric traveling salesman problem (ATSP).

   (b) Give an IP formulation of the problem.

   (c) Use branch-and-bound to solve the IP to optimality. Draw the search tree indicating the branching variables and the LP lower bounds.

(3 pts)

2. Let $G = (V, E)$ be an undirected graph and M a matching in G. A (simple) path in G is called an M-alternating path if it consists alternately of edges belonging to M and not, like in the figure (the bold edges are matching edges).



An M-alternating path between two M-exposed nodes u and v is called an M-*augmenting* path. Show that M is a matching of maximum cardinality if and only if there are no M-augmenting paths in G.

(4 pts)

3. Given and undirected graph $G = (V, E)$, with $n := |V|$, we define the polytope

$$P := \left\{ x \in [0,1]^E : \sum_{e \in E} x_e = n - 1, \sum_{e \in \delta(X)} x_e \geq 1, \text{for } \emptyset \subset X \subset V \right\}$$

Show that P *strictly* contains the spanning tree polytope of G, i.e., the convex hull of all incidence vectors of spanning trees.

(3 pts)