Packing Problems with Nonlinear Optimization

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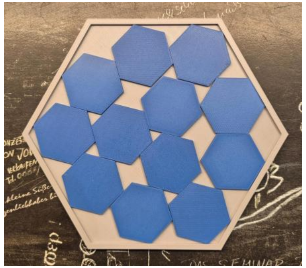
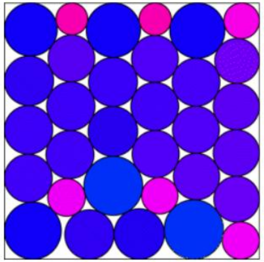
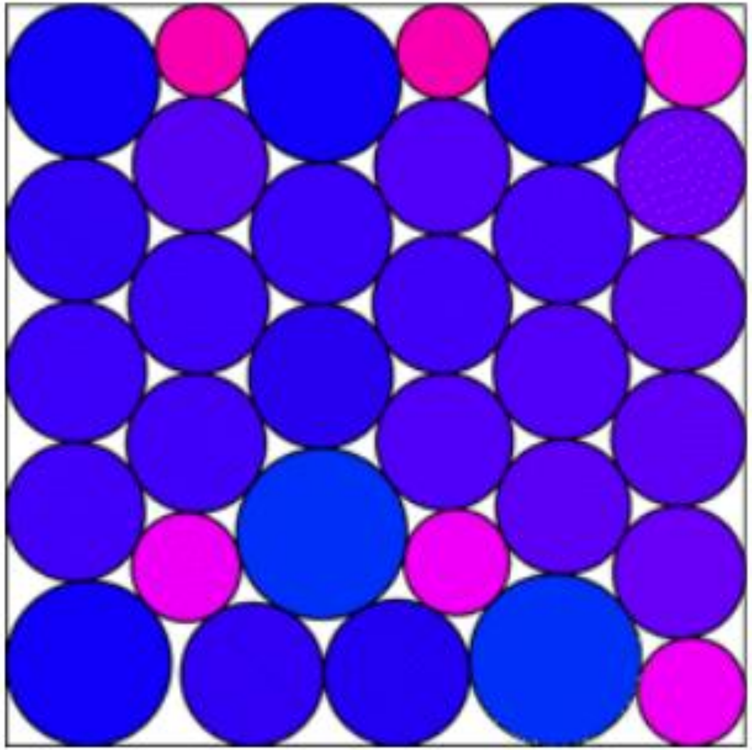
How to Get Material for a Paper in One Afternoon

The Motivation

Packing Problems


- In the intersection of recreational Math and highly relevant business problems
  - There are good heuristics and relevant Math Programming approaches
  - …and there are benchmarks documenting records for best circle, polygon, etc. packings
    - Packing Center: https://erich-friedman.github.io/packing/



Breaking News!!! Deepmind-AI Solves Math Problems

- News from May 2025

The Economic Times | tech

English Edition | Today's ePaper

Tech › Tech & Internet › Google DeepMind's new AI coding tool can solve complex math problems, design algorithms

Google DeepMind's new AI coding tool can solve complex math problems, design algorithms

nature

nature › news › article

NEWS | 14 May 2025 | Correction 15 May 2025

DeepMind unveils 'spectacular' general-purpose science AI

NewScientist

Mathematics

Mathematicians say Google's AI tools are supercharging their research

AlphaEvolve, an AI system created by Google DeepMind, is helping mathematicians do research at a scale that was previously impossible - even if it does occasionally "cheat" to find a solution

By Alex Wilkins

18 November 2025

IEEE Spectrum FOR THE TECHNOLOGY INSIDER

NEWS | AI

New AI Model Advances the "Kissing Problem" and More › AlphaEvolve made several mathematical discoveries and practical optimizations

BY DINA GENKINA | 14 MAY 2025 | 4 MIN READ

INDIA TODAY

News / Technology / News / Google DeepMind brings AlphaEvolve tuned to tackle maths a…

Google DeepMind brings AlphaEvolve, an AI tuned to tackle maths and computing problems

Reminder: What Happened?

- Google Deepmind published a blog: new solutions for 14 Math problems
  - Found with AlphaEvolve Coding AI
  - Including various packing problems
- One algorithm trained per problem
  - Presumably over many hours or even days, when including evaluation?
- Updated in November 2025 with new results for 67 problems
  - Further updates in May 2026

May 14, 2025 Science

AlphaEvolve: A Gemini-powered coding agent for designing advanced algorithms

AlphaEvolve team

Share

Alphaevolve: A coding agent for scientific and algorithmic discovery
A Novikov, N Vũ, M Eisenberger, E Dupont… - arXiv preprint arXiv …, 2025 - arxiv.org
In this white paper, we present **AlphaEvolve**, an evolutionary coding agent that substantially enhances capabilities of state-of-the-art LLMs on highly challenging tasks such as tackling …
☆ Speichern 99 Zitieren Zitiert von: 614 Ähnliche Artikel Alle 3 Versionen

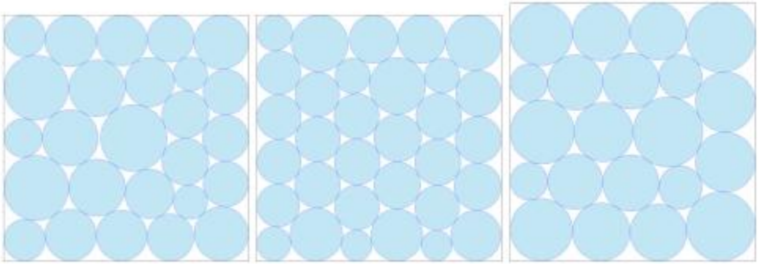


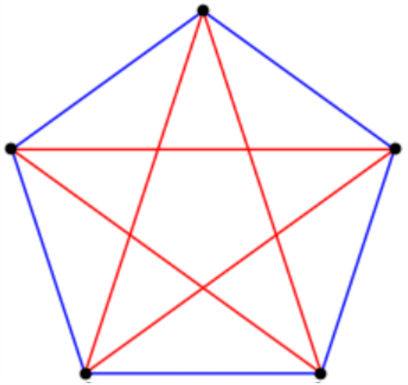
Figure 14 | New constructions found by *AlphaEvolve* improving the best known bounds on packing circles to maximize their sum of radii. Left: 26 circles in a unit square with sum

Image source: Google Deepmind
https://deepmind.google/blog/alphaevolve-a-gemini-powered-coding-agent-for-designing-advanced-algorithms/

First results: Min-Max ratios

For Completeness: Min-Max Ratio Problems

- Problem: Find configuration of points that is as evenly spaced as possible
  - Minimize ratio between the maximum and the minimum pairwise distance
  - Modeling trick: Fix minimum to 1, minimize maximum distance
  - Modeling trick: minimize squared distance to avoid roots
- Variables: point coordinates $x_{i,k}$ with $k \in D$ the k-th of $|D|$ dimensions
- Constraints:
  - All (squared) distances greater equal one: $\sum(x_{i,k} - x_{j,k})^2 \geq 1$
  - All (squared) distances less equal maximum: $\sum(x_{i,k} - x_{j,k})^2 \leq \mathrm{t}_{\mathrm{max}}$
- Objective: $min\ t_{max}$
  - Could formulate a dual where you fix the max and maximize the min

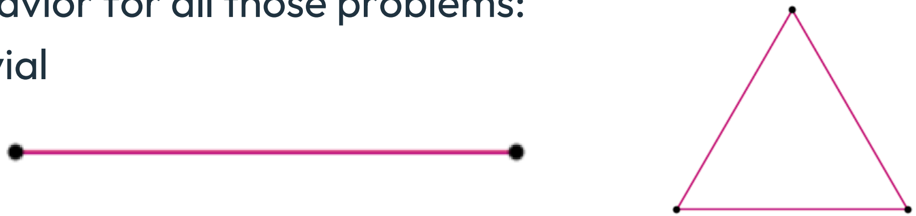


$$
\begin{aligned}
&\min\ t_{\max} \\
\text{s.t.}\quad & \sum_{k \in \mathcal{D}} (x_{i,k} - x_{j,k})^2 \geq 1, \\
& \sum_{k \in \mathcal{D}} (x_{i,k} - x_{j,k})^2 \leq t_{\max}, \\
& t_{\max} \geq 1
\end{aligned}
$$

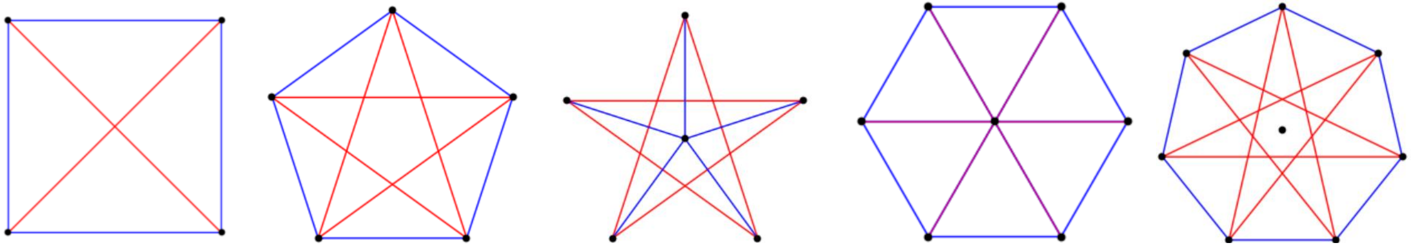
For Completeness: Min-Max Ratio Problems

Typical behavior for all those problems:

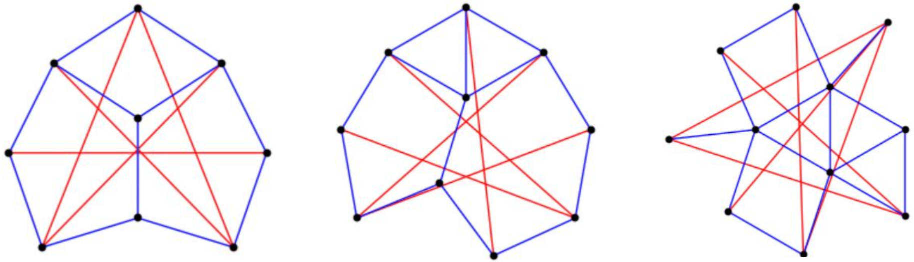
- Starts trivial



- Continues regular



- At some point, it gets funny



New Best Solutions

d	n	Squared distance ratio	Previous best	Source
2	16	12.8892**4**	12.88927	[19,25]
2	21	17.77**499**	17.776	[16]
2	22	19.05**398**	19.055	[16]
2	29	25.92**460**	25.929	[1]
3	14	4.165**78**	4.16585	[19,25]

Many known solutions recovered

Some new and improved solutions found

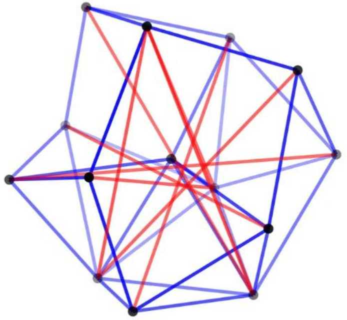
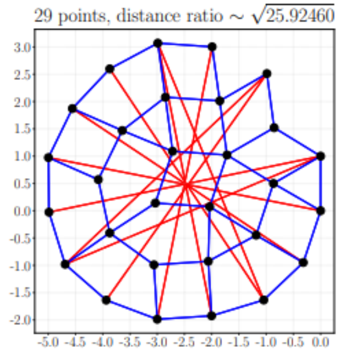
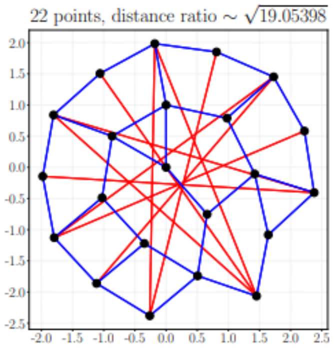
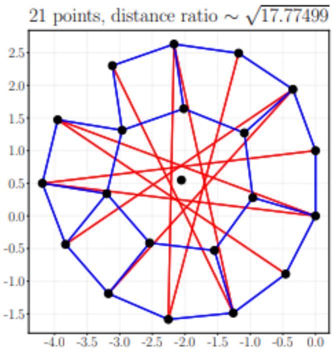
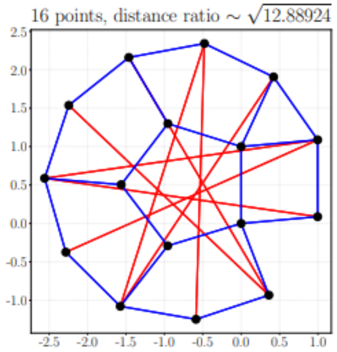


Figure 2. The new solution to the 14-point 3D distance ratio problem.







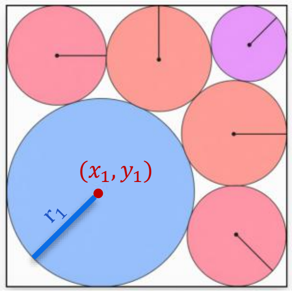


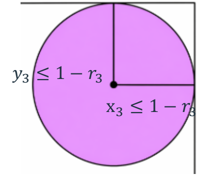
Circle packing

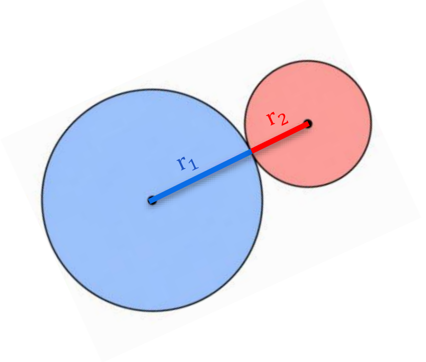
Optimization Model for Circle Packing

- Problem: place n circles in a unit square/rectangle of perimeter 4 to maximize the sum of their radii
- Variables: circle centers x_i and y_i, radius r_i
- Constraints:
  - Each circle fully inside the rectangle:
    - $r_i \leq x_i \leq \alpha - r_i, r_i \leq y_i \leq (2-\alpha) - r_i$
  - Circles non-overlapping (center distance greater than the sum of radii):
    - $(x_1 - x_2)^2 + (y_1 - y_2)^2 \geq (r_1 + r_2)^2$
- Objective: $\max \sum r_i$
- Easy to change from rectangle of perimeter 4 to unit square (set $\alpha = 1$)

$$
\begin{aligned}
\max \quad & \sum_{i \in \mathcal{N}} r_i \\
\text{s.t.} \quad & r_i \leq x_i \leq \alpha - r_i, \\
& r_i \leq y_i \leq (2-\alpha) - r_i, \\
(x_i - x_j)^2 + (y_i - y_j)^2 &\geq (r_i + r_j)^2, \\
& 0 \leq r_i \leq \tfrac{\alpha}{2}, \\
& 0 < \alpha \leq 1
\end{aligned}
$$





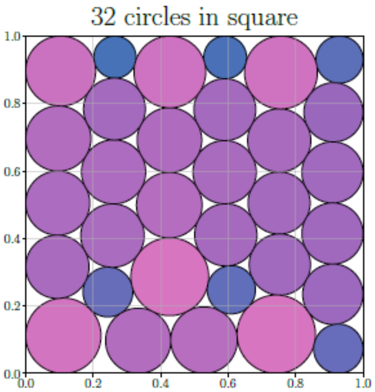


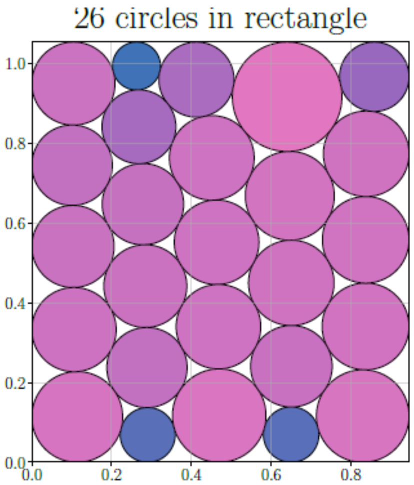
New Packings

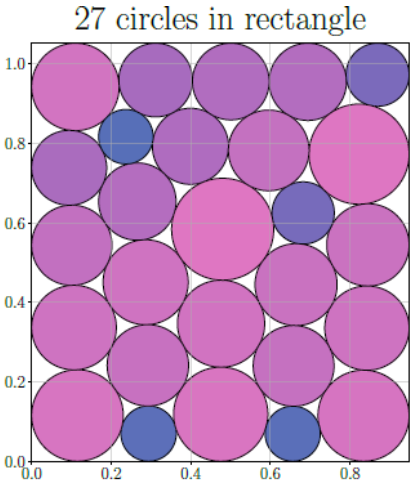
Variant	n	Sum of radii	Previous best
square	32	2.93**957**	2.93794
rectangle	26	2.63**930**	2.638
rectangle	27	2.6**9015**	2.687

All known solutions recovered

Some new and improved solutions found







Area Inequalities

- This model is very weak

$$\sum_{i \in \mathcal{N}} r_i \le \sqrt{n} \sqrt{\sum_{i \in \mathcal{N}} r_i^2} \le \sqrt{n} \sqrt{\frac{\alpha(2-\alpha)}{\pi}} \le \sqrt{\frac{n}{\pi}}$$

- Typical dual bound from the solvers is n/2.
- Trivial(?) area inequality

$$\sum_{i \in \mathcal{N}} r_i^2 \le \frac{1}{\pi}$$

- Leads to better performance
- This will be used for all the problems in this talk

Symmetries and how to avoid them

All of these problems have two levels of symmetry

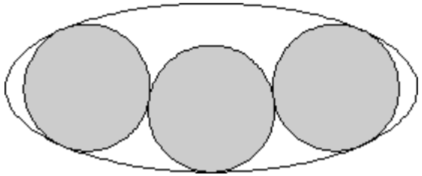
1. The arrangement can be mirrored, rotated, etc.
   - Choose a canonical representative for each symmetric solution
2. The objects can be renumbered ($n!$)
   - Sort them in a fixed way (left to right, or along an arbitrary line)
   - Also helps in controlling the quadratic growth of non-overlap constraints

- Symmetries matter for the dual bound, but have less impact on the primal solution
  - Only minimal symmetry breaking was used for this project

Packing Unit Circles in an Ellipse

- Problem: Pack *n* unit circles in an ellipse of minimum area

- Solutions exist for up to 20 circles since 2007.



- Place the ellipse centred at the origin.
- Variables:
  - x_i, y_i: center of circle i
  - a, b: semiaxes of the ellipse

- Circle separation is as before
  - How about circle containment?

Packing Unit Circles in an Ellipse

The circle-ellipse inclusion

- **Theorem**: A circle of radius r centred at (x, y) is contained in an ellipse centred at the origin with semiaxes a and b if and only if there is a multiplier t such that the matrix $\begin{bmatrix} A & B & 0 \\ B & E & D \\ 0 & D & C \end{bmatrix}$

  is positive semidefinite, where

$$\begin{aligned} A &= t - b^2 \\ B &= -tx_0 \\ C &= t - a^2 \\ D &= -ty_0 \\ E &= a^2b^2 + t\left(x_0^2 + y_0^2 - r^2\right) \end{aligned}$$

- The PSD condition is then expressed using the nonnegativity of the principal minors.

- **Proof**: Using the S-lemma!

$$\nexists x, y$$

Circle inside ellipse $\longleftrightarrow$ $(x - x_0)^2 + (y - y_0)^2 \leq r^2$ $\longleftrightarrow$ $\exists t \geq 0 : \forall x, y \in \mathbb{R} : a^2b^2 - b^2x^2 - a^2y^2 + t\left((x - x_0)^2 + (y - y_0)^2 - r^2\right) \geq 0$

$$b^2x^2 + a^2y^2 > a^2b^2$$

- Useful: Know the literature/theorems
  - In this case: Have the original author of the theorem on your team☺

Packing Unit Circles in an Ellipse: Complete Model

$$\min \; a \cdot b \cdot \pi$$

$$\text{s.t.} \quad \boxed{(x_i - x_j)^2 + (y_i - y_j)^2 \geq 4,}$$

Circles are disjoint

Model has degree 8

$$
\boxed{
\begin{aligned}
t_i - a^2 &\geq 0, \\
t_i - b^2 &\geq 0, \\
a^2 b^2 + t_i(x_i^2 + y_i^2 - 1) &\geq 0, \\
(t_i - a^2)(a^2 b^2 + t_i(x_i^2 + y_i^2 - 1)) - t_i^2 y_i^2 &\geq 0, \\
(t_i - b^2)(a^2 b^2 + t_i(x_i^2 + y_i^2 - 1)) - t_i^2 x_i^2 &\geq 0, \\
(t_i - a^2)(t_i - b^2)(a^2 b^2 + t_i(x_i^2 + y_i^2 - 1)) & \\
- t_i^2(x_i^2(t_i - a^2) + y_i^2(t_i - b^2)) &\geq 0,
\end{aligned}
}
$$

Circles are in the ellipse

$$
\boxed{
\begin{aligned}
t_i &\geq 1, \\
a, b &\geq 1
\end{aligned}
}
$$

Variable bounds

$$\boxed{a \geq b}$$

Symmetry breaking

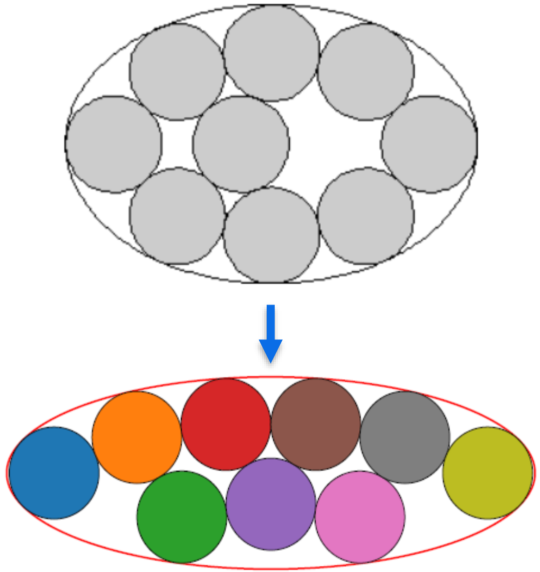
Strengthening the model

- Area constraint: $ab \geq n$
  - This also implies $a \geq \sqrt{n}$
- Circle centers are inside the ellipse $b^2 x_i^2 + a^2 y_i^2 \leq a^2 b^2$
- Circles are far away from the "ends" of the ellipse

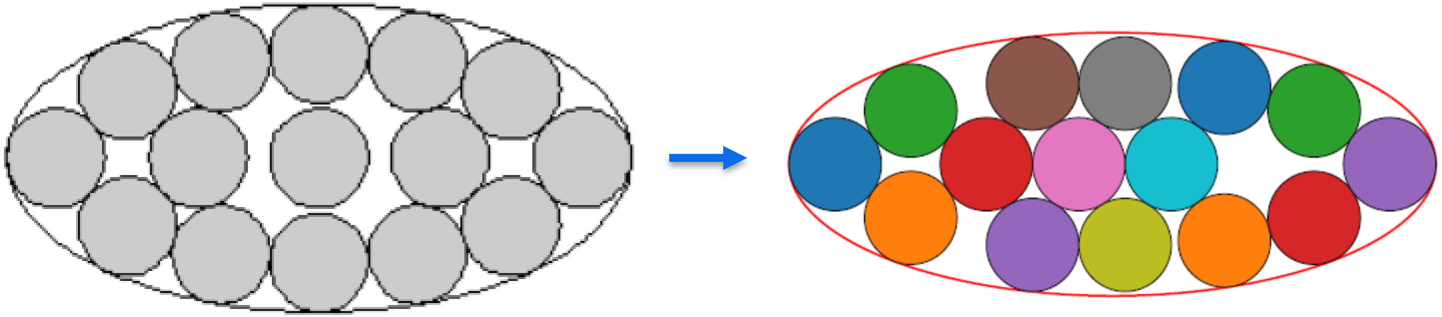
$$x_i \leq a - 1$$
$$x_i \geq -a + 1$$
$$y_i \leq b - 1$$
$$y_i \geq -b + 1$$

New Packings

- All known solutions recovered
- Some new and improved solutions found
  - Different configurations, not just slight improvements



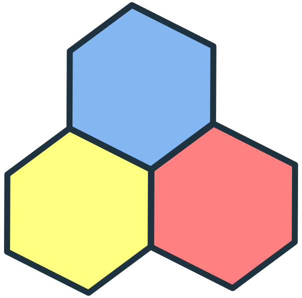
Size	New objective	Old objective
9	12.2708π	12.403π
15	19.76329π	19.824π

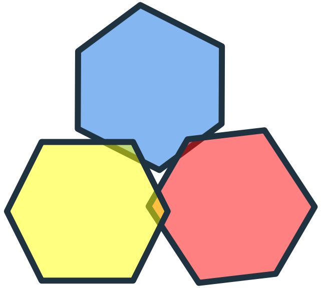


Polygon packing

Polygon Packing

- Problem: pack n regular m-gons into the smallest regular ℓ -gon
- Very well studied for squares into square and triangles into triangles
- Challenge for modelling: Nonoverlapping circles are easy…
  - (shape literally defined equidistance, perfect rotational symmetry)
- …, however, nonoverlapping polygons are harder, rotation comes into play
- Intersection between polygons empty?
  - i.e., a system of inequalities does not have a solution?
  - And we want to model that by a different system having a solution?
  - Farkas for the rescue!





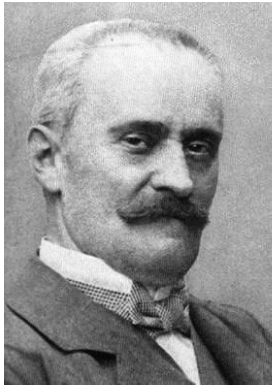


Image source: Wikipedia

An Application of the Farkas Lemma

Lemma 3 Two convex m-gons with half-space representations $\{a_{i,k}x + b_{i,k}y \geq s_{i,k}\}_{k\in\mathcal{M}}$ and $\{a_{j,k}x + b_{j,k}y \geq s_{j,k}\}_{k\in\mathcal{M}}$ have disjoint interiors if and only if there exist nonnegative multipliers $\lambda_1, \ldots, \lambda_{2m}$ summing to one with

$$\sum_{k\in\mathcal{M}} \lambda_{k+1}a_{i,k} + \sum_{k\in\mathcal{M}} \lambda_{k+m+1}a_{j,k} = 0 \tag{6.1a}$$

$$\sum_{k\in\mathcal{M}} \lambda_{k+1}b_{i,k} + \sum_{k\in\mathcal{M}} \lambda_{k+m+1}b_{j,k} = 0 \tag{6.1b}$$

$$\sum_{k\in\mathcal{M}} \lambda_{k+1}s_{i,k} + \sum_{k\in\mathcal{M}} \lambda_{k+m+1}s_{j,k} \geq 0 \tag{6.1c}$$

i.e., a canceling normal combination with nonnegative gap.

Polygon Packing: Variables

- R: circumradius of the outer ℓ-gon (to be minimized)
- x_i, y_i: centers of the inner m-gons
- θ_i: rotation angles of the inner m-gons ($\in [0, {}^{2\pi}/_{m}]$)
- $a_{i,j}, b_{i,j}$: components of edges of inner m-gons
- $s_{i,j}$: offsets of edges of inner m-gons
- $\lambda_{i,j,k}$: Farkas multipliers

Polygon Packing: The Full Model

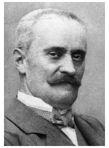
$$\min \ R$$

All inner vertices within outer ℓ-gon

$$\begin{aligned} R\rho_\ell + \sin(k\phi_\ell)\big(x_i + \sin(\theta_i + \delta_{m,j})\big) \\ + \cos(k\phi_\ell)\big(y_i + \cos(\theta_i + \delta_{m,j})\big) \geq 0, \end{aligned}$$

Inner representation matches outer representation

$$\begin{aligned} a_{i,j} &= \sin(\theta_i + j\phi_m), \\ b_{i,j} &= \cos(\theta_i + j\phi_m), \\ s_{i,j} &= a_{i,j}x_i + b_{i,j}y_i - \rho_m, \end{aligned}$$



polygons don't intersect

$$\sum_{k=1}^{2m} \lambda_{i,j,k} = 1,$$

$$\sum_{k=0}^{m-1} \lambda_{i,j,k+1} a_{i,k} + \sum_{k=0}^{m-1} \lambda_{i,j,k+m+1} a_{j,k} = 0,$$

$$\sum_{k=0}^{m-1} \lambda_{i,j,k+1} b_{i,k} + \sum_{k=0}^{m-1} \lambda_{i,j,k+m+1} b_{j,k} = 0,$$

$$\sum_{k=0}^{m-1} \lambda_{i,j,k+1} s_{i,k} + \sum_{k=0}^{m-1} \lambda_{i,j,k+m+1} s_{j,k} \geq 0,$$

symmetry breaking

$$0 \leq \theta_i \leq \phi_m,$$

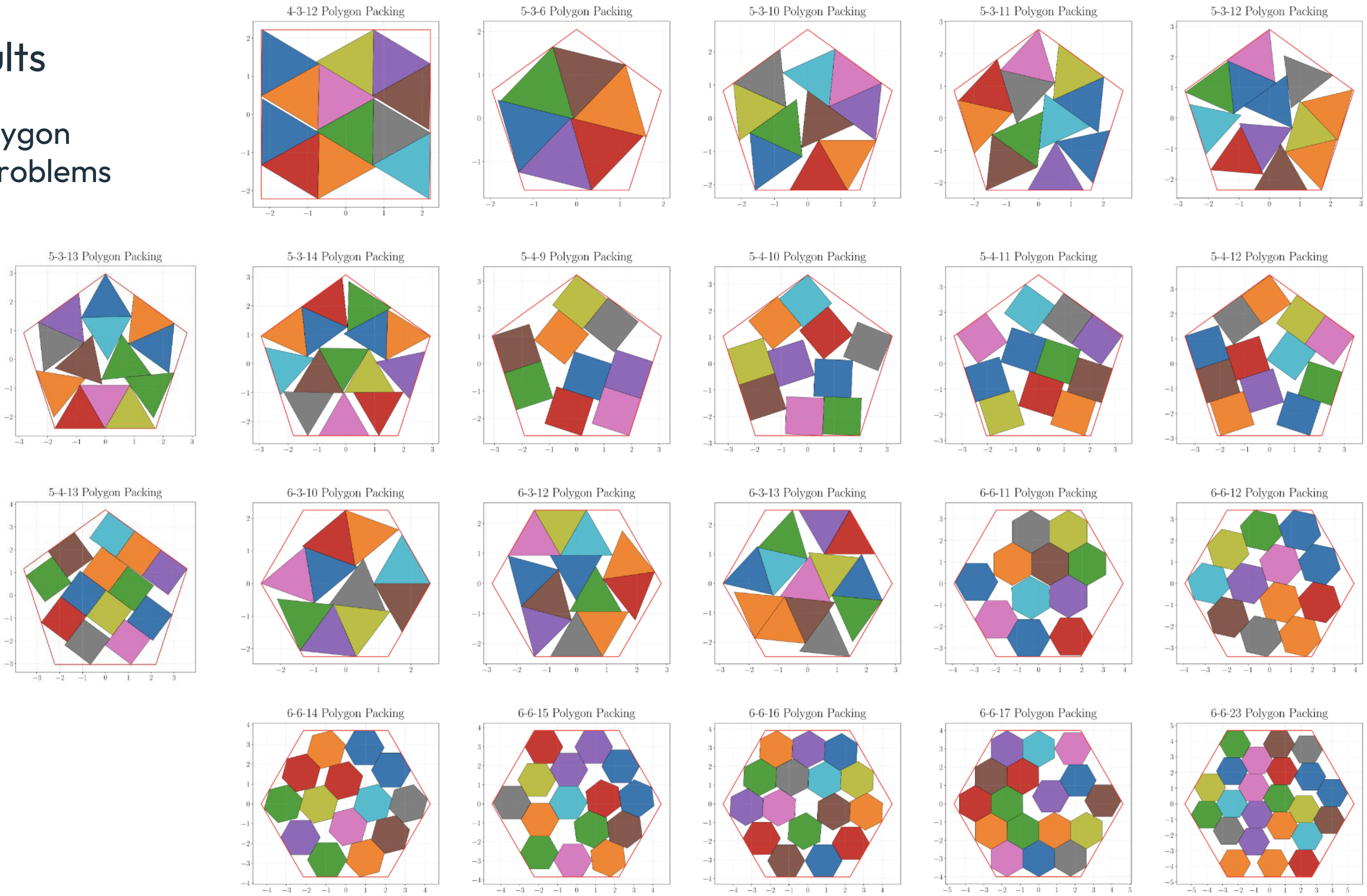
$$\lambda_{i,j,k} \geq 0,$$

redundant, but dual bound-strengthening

$$\begin{aligned} (x_i - x_j)^2 + (y_i - y_j)^2 \geq (2\rho_m)^2, \\ R \geq R_{\min} \end{aligned}$$

New Results

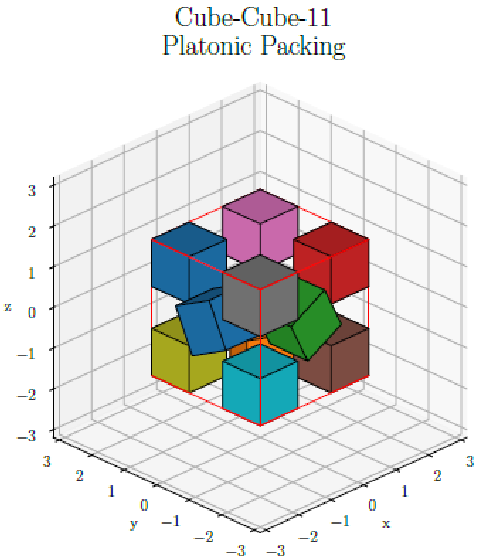
- For 22 polygon packing problems

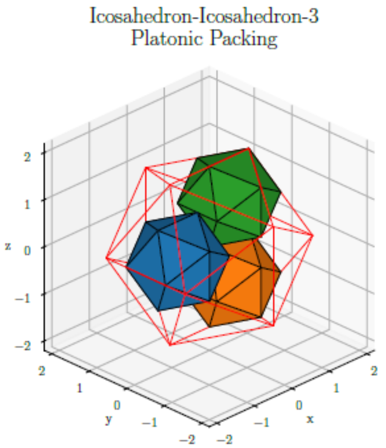


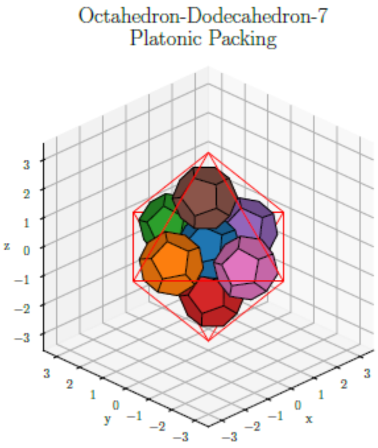
New(ish): Platonic Solid Packing

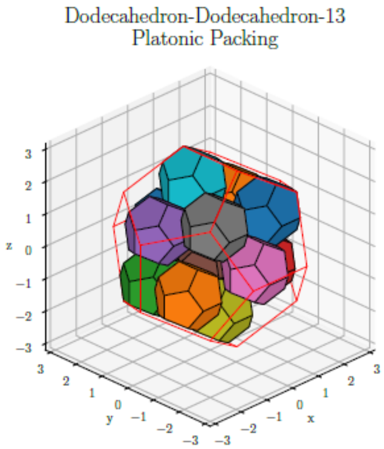
The Model Generalizes to Higher Dimensions (Different Shapes, More Objects...)

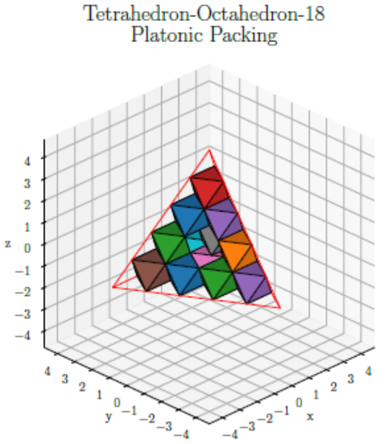
- We introduced the problem of packing Platonic solids into Platonic solids
- Important special case: Cubes into cubes (already well-studied)
  - Yet another new record

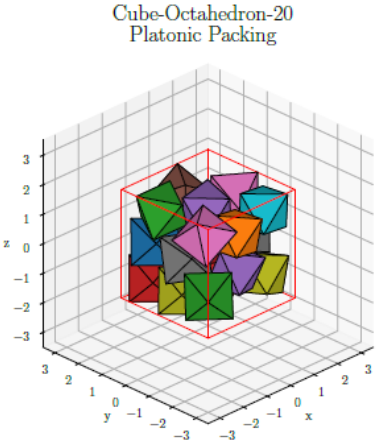


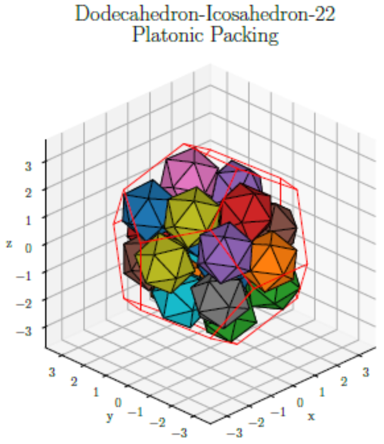












Some Comments

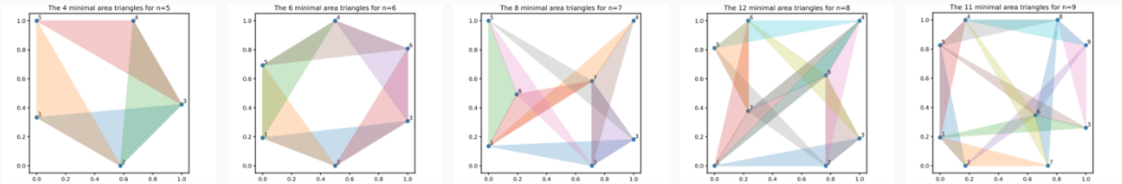
- All of our solutions have been polished
  - They are feasible to machine precision
  - See: Numerics lecture
- None of these solutions are “proved” optimal
  - Special bounding tightening and cutting techniques is needed from the solvers
  - Some papers exist, could be implemented
- One could get exact coordinates for all of these solutions using polynomial systems and Gröbner bases
  - This has been done to the Heilbronn problem in a recent paper

The Heilbronn problem

The Heilbronn problem

- Another beautiful (and relatively straightforward) application of nonlinear optimization

How should n points be placed in the unit square s.t. the smallest triangle determined by any three of them is as large as possible?



$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$\Delta_5 = \frac{\sqrt{3}}{9}$	$\Delta_6 = \frac{1}{8}$	$\Delta_7 = f - \frac{1}{2}$	$\Delta_8 = \frac{\sqrt{13}-1}{36}$	$\Delta_9 = \frac{9\sqrt{65}-55}{320}$

All proved optimal. Exact coordinates recovered by the optimize-then-refine framework. Note that the optima need *not* be symmetric, see $n = 7$.

Next frontier: can we compute and certify the optimum for $n = 10$?

- See [Sudermann-Merx 2026]. Slide and video by Nathan. Updates coming soon ☺

Impact

Our Message

- Global nonlinear optimization has become a “model&solve” technology
  - Off-the-shelf Xpress simply solves problems other approaches struggle with
  - Setting 35 records on packing problems (and counting)
- General solvers (and models) are much more versatile than hand-tuned algorithms
  - We suspect days/weeks of training
  - Dual certificates are not very strong here, but they “come for free”
- LLM results are impressive, enable new user groups
  - “Claude, write me a model for …” and plugging that into Xpress would have worked well for circle-packing

- Question: How do we make genAI optimization-aware?

Global Optimization Beats GenAI

- Very successful Blog Post: https://www.fico.com/blogs/best-global-optimization-solver
  - „Viral“ LinkedIn post with 82k impressions
- Covered in two German newspapers and in OR/MS Today
  - Each with 5-digit circulation (total 80k)
- Two scientific papers written
  - One published in ISCO proceedings, the other submitted for JOGO
  - Third paper on Heilbronn problem in preparation

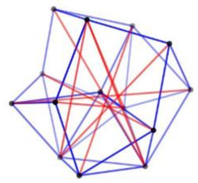
1.096

82.774 Impressions

Timo Berthold • Sie
Mathematical Optimization Researcher, Teacher, and Director
9 Monate •

FICO Xpress Optimization Surpasses Google DeepMind's State-of-the-Art Achievements with Mathematical Optimization
You may have heard the news that #AlphaEvolve by Google #Deepmind recently set new records for 13 unsolved Math problems? We were curious—what happens if we tackle some of these with mathematical optimization instead of AI heuristics? Using the FICO Xpress Solver, we modeled several of the same problems as global optimization instances—and in many cases, we didn't just match AlphaEvolve's solutions... we found better ones! Learn how we did that in my latest blog: https://lnkd.in/dx2MzcT4
I'm genuinely excited about this work and proud of what our team (Imre Pólik, Gioni Mexi, Bruno Vieira, Susanne Heipcke, Liding Xu) has achieved. Wishing you all a great rest of your week!

Übersetzung anzeigen



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By Timo Berthold, Imre Pólik

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Timo Berthold
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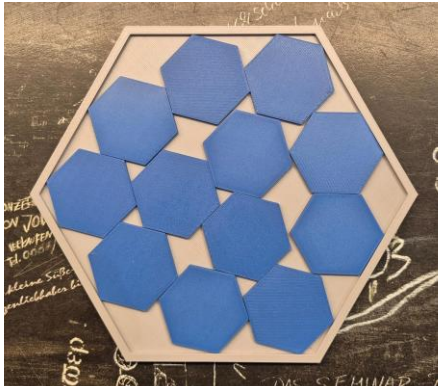
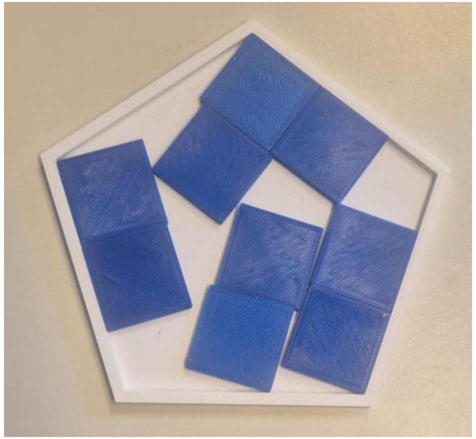
Outreach: Help people get hands-on-experiences with Optimization. Literally!

- You can 3D-print and play with your Optimization solutions (thanks to Philipp and Anita!)
  - Well-received at Tag der Mathematik, Lange Nacht der Wissenschaften, Advisory Board visit, Conferences





Thank You!

PD Dr. Timo Berthold
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