COMPUTATIONAL INTEGER OPTIMIZATION · LECTURE 12

Conflict Analysis in MIP

Learning from Infeasibility

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$$\min \left\{ c^{\mathsf{T}} x \;:\; Ax \geq b,\;\; \ell \leq x \leq u,\;\; x \in \mathbb{Z}^{\mathcal{I}} \times \mathbb{R}^{\mathcal{C}} \right\}$$

You have already seen this week:

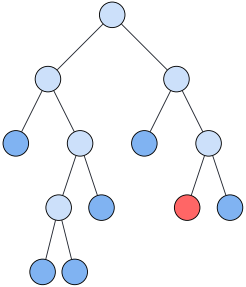
- ◇ branch on fractional variables, solve LP relaxations, prune by bound or infeasibility
  - propagation can make a constraint infeasible
  - the LP relaxation is infeasible or exceeds the cutoff bound

Goal of conflict analysis

◇ Given an infeasible node, e.g.,

$x_1 = 1 \rightarrow x_2 = 1 \rightarrow x_3 = 1 \rightarrow x_4 = 0 \rightarrow x_5 = 1 \rightarrow x_6 = 1 \rightarrow \perp$

∗branching ∗propagation ∗conflict



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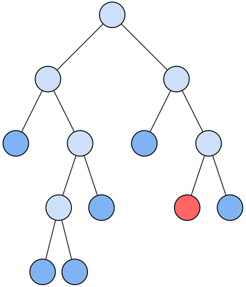
$*$branching $*$propagation $*$conflict

- find an explanation for the infeasibility



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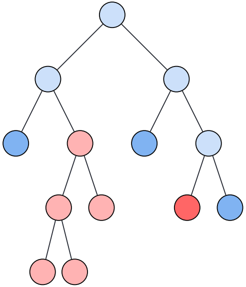
- find an explanation for the infeasibility
- prune other parts of the tree



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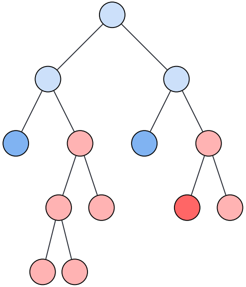
$*$branching $*$propagation $*$conflict

- find an explanation for the infeasibility
- prune other parts of the tree
- prune in backtracking

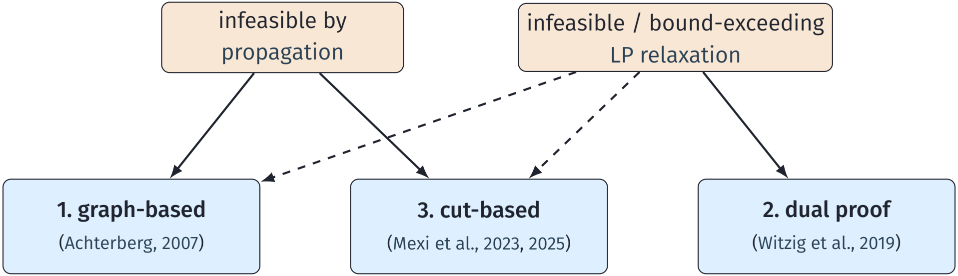


How much is a solver component worth?

349 instances $\times$ 5 seeds, 2 h limit. Ratio: shifted geom. mean time vs. default. Consistent winners: $\geq$10 % faster on all five seeds.

			Consistent Winners	
Configuration	Ratio	Solved	Default	Config
Default SCIP	—	1462	—	—
No conflict analysis	**1.31**	**1361**	**40**	**2**

- Without conflict analysis: 31 % slower, -101 instances solved
- and almost never harmful: only 2 consistent wins for the no-conflict configuration



1. graph-based: SAT-style analysis of the implication graph, learns bound disjunctions
2. dual proof: aggregate rows with the Farkas certificate of the infeasible LP
3. cut-based: resolve the linear constraints themselves, strengthened by cuts

Origins: Conflict Analysis in SAT

SAT stands for (propositional) satisfiability problems

- binary variables x_i, literals x_i or $\bar{x}_i = \neg x_i$
- clauses, e.g., $x_1 \vee x_2 \vee \bar{x}_3$; goal: find a feasible assignment

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For example a Sudoku: let p_{ijk} be `True` iff digit k goes in cell (i, j)

- must place a digit in (i, j): $\quad p_{ij1} \vee p_{ij2} \vee \ldots \vee p_{ij9}$
- at most one digit in (i, j): $\quad \bar{p}_{ijk} \vee \bar{p}_{ijl} \quad \forall k \neq l,$ $\quad$ etc.

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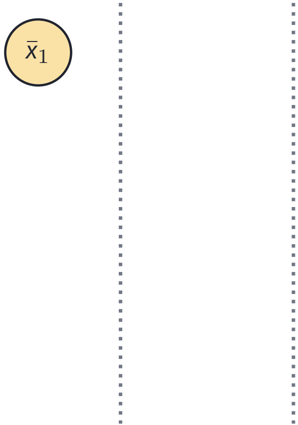
How a CDCL SAT solver works (very simplified):

- branching: fix a literal
- propagation: in $x_1 \vee x_2 \vee \bar{x}_3$, if $x_1 = \mathrm{F}$ and $x_3 = \mathrm{T}$, then $x_2 = \mathrm{T}$
  (all but one literal falsified $\Rightarrow$ the last one must be `True`)
- conflict analysis: when propagation runs into a contradiction, learn from it

$$
\begin{aligned}
C_1 &: x_1 \vee \bar{x}_2 \\
C_2 &: \bar{x}_3 \vee x_4 \\
C_3 &: x_2 \vee \bar{x}_4 \vee x_5 \\
C_4 &: \bar{x}_5 \vee \bar{x}_6 \\
C_5 &: \bar{x}_5 \vee \bar{x}_7 \vee x_8 \\
C_6 &: \bar{x}_7 \vee x_9 \\
C_7 &: \bar{x}_5 \vee \bar{x}_8 \vee \bar{x}_9 \vee x_{10} \\
C_8 &: x_6 \vee \bar{x}_{10} \vee x_{11} \\
C_9 &: \bar{x}_{10} \vee x_{12} \\
C_{10} &: \bar{x}_{11} \vee \bar{x}_{12}
\end{aligned}
$$

Solving process



$c_1 : x_1 \vee \bar{x}_2$

$c_2 : \bar{x}_3 \vee x_4$

$c_3 : x_2 \vee \bar{x}_4 \vee x_5$

$c_4 : \bar{x}_5 \vee \bar{x}_6$

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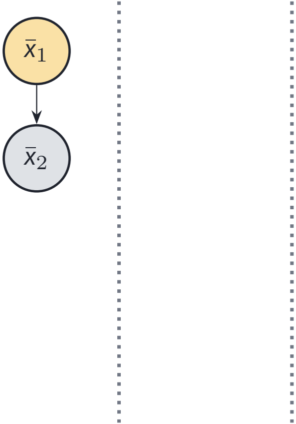
Solving process



$c_1 : \boxed{x_1 \vee \bar{x}_2}$

$c_2 : \bar{x}_3 \vee x_4$

$c_3 : x_2 \vee \bar{x}_4 \vee x_5$

$c_4 : \bar{x}_5 \vee \bar{x}_6$

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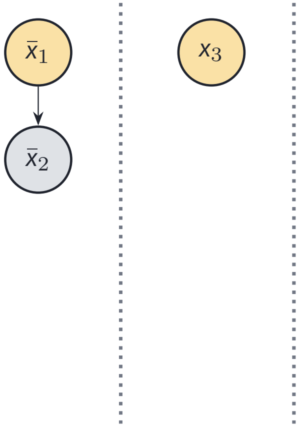

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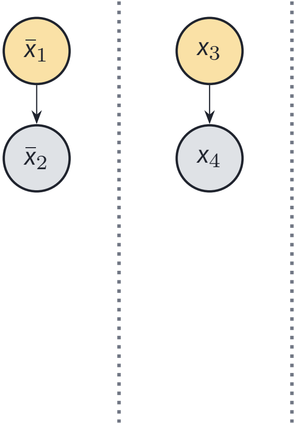

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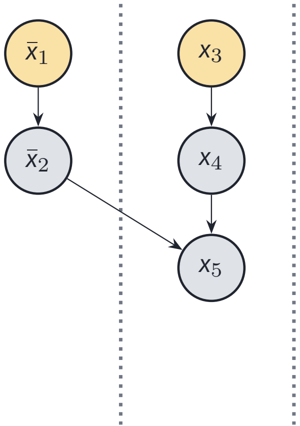

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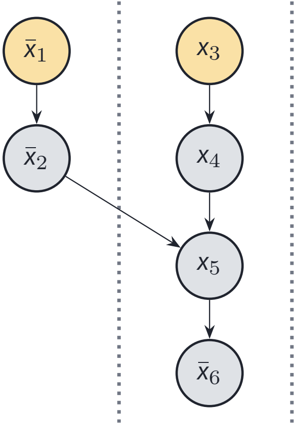

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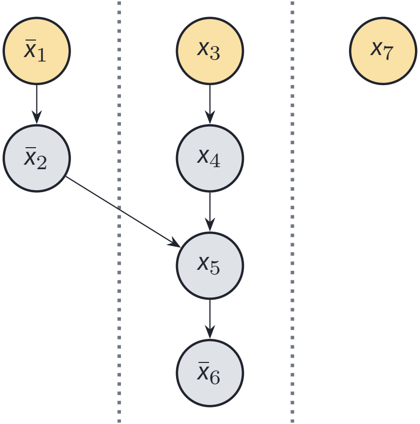

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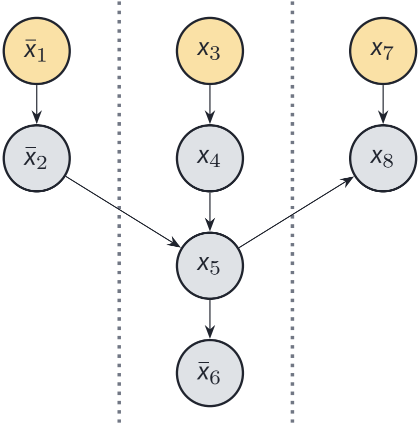

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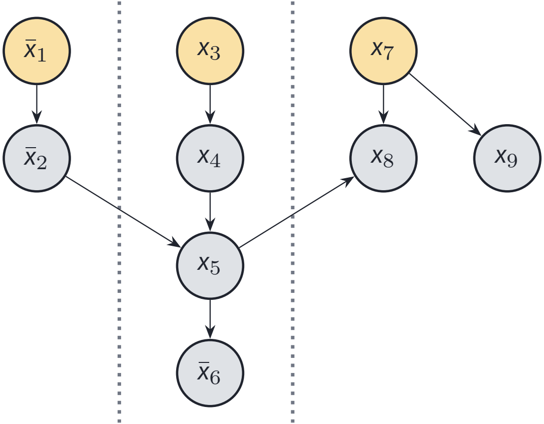

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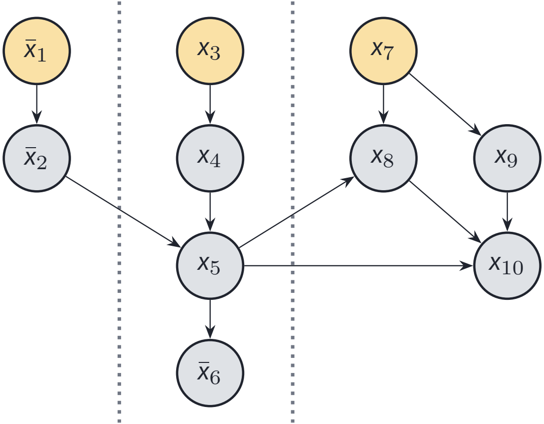

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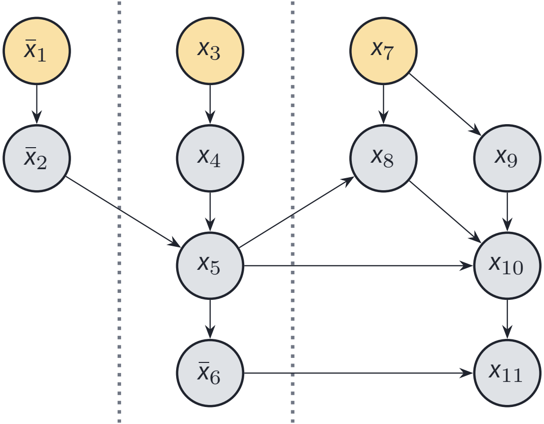

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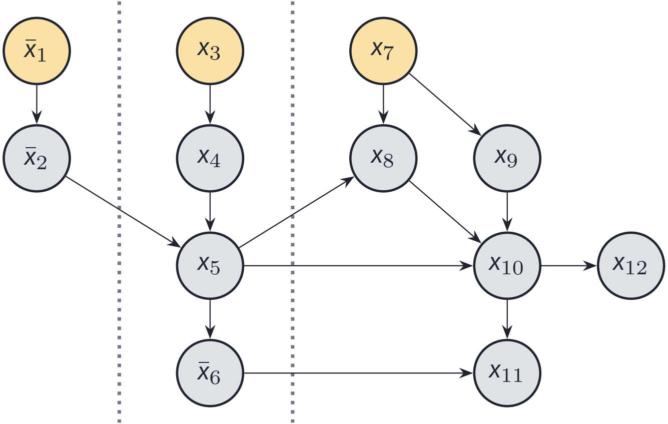

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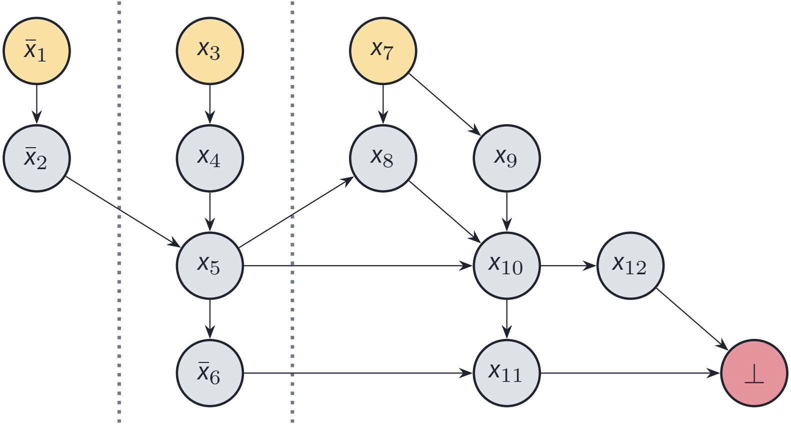

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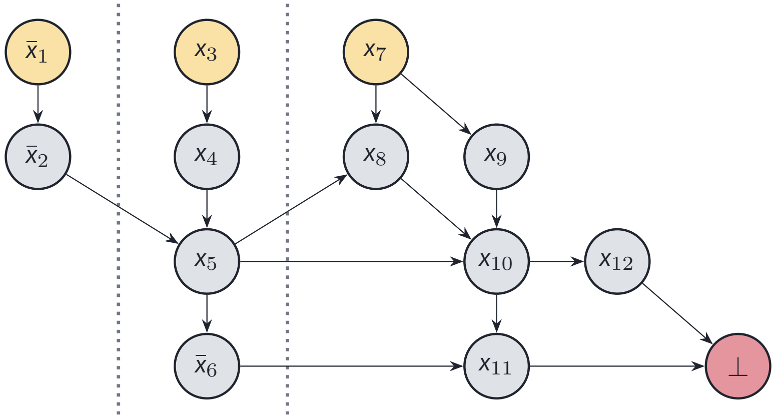
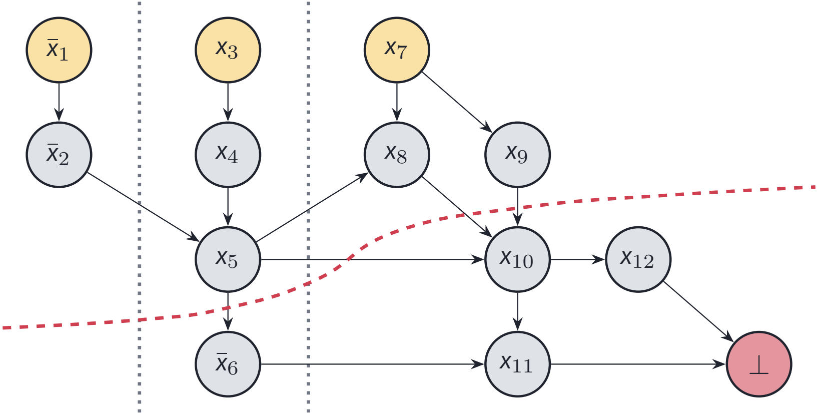
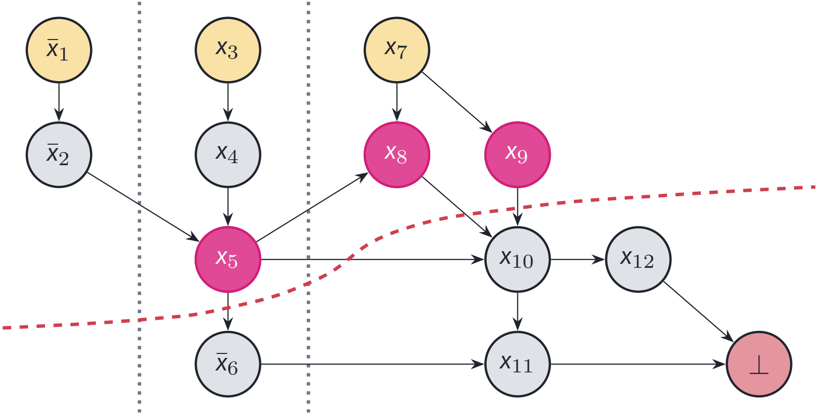
Conflict analysis: cuts in the implication graph





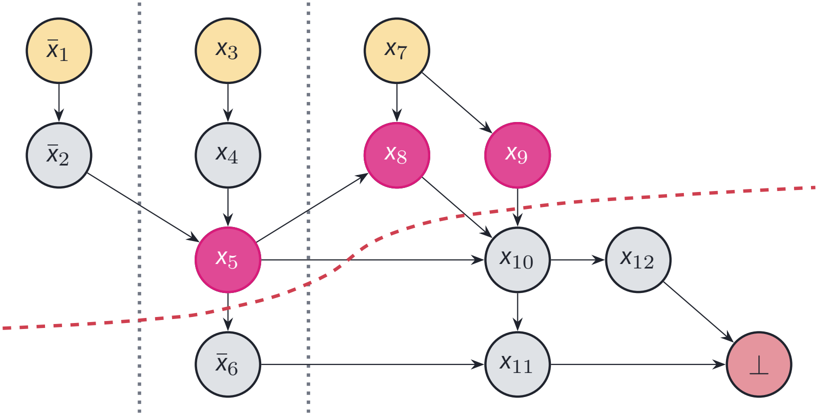
Key insight: any cut (in the graph sense) separating the branching decisions from the conflict node yields a set of literals of which at least one must be `False`.

Conflict analysis: cuts in the implication graph

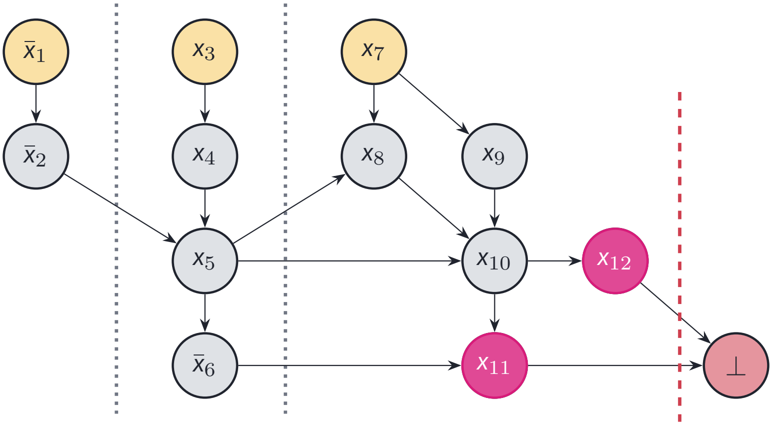


Key insight: any cut (in the graph sense) separating the branching decisions from the conflict node yields a set of literals of which at least one must be `False`.

Conflict analysis: which cut to choose?

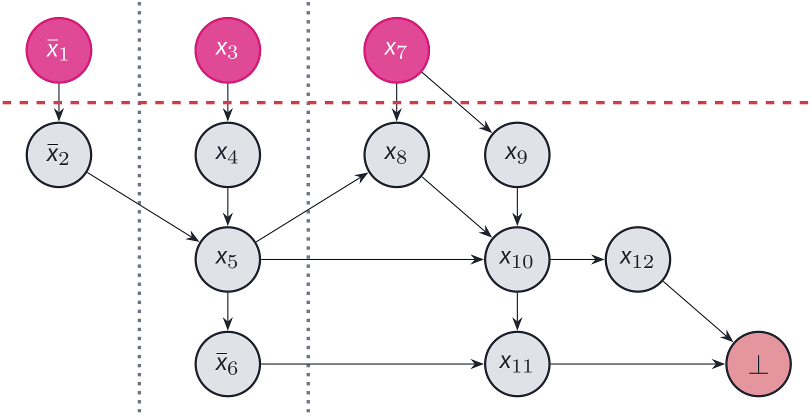


$$\bar{x}_5 \vee \bar{x}_8 \vee \bar{x}_9$$

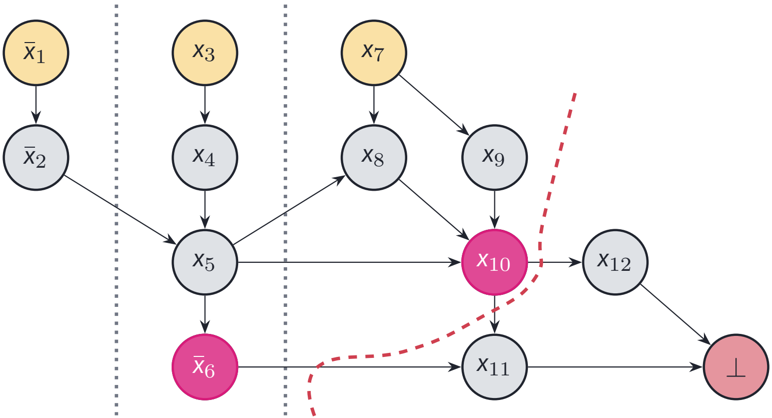


$\bar{x}_{11} \vee \bar{x}_{12}$ (C_{10} itself: useless)

Conflict analysis: which cut to choose?

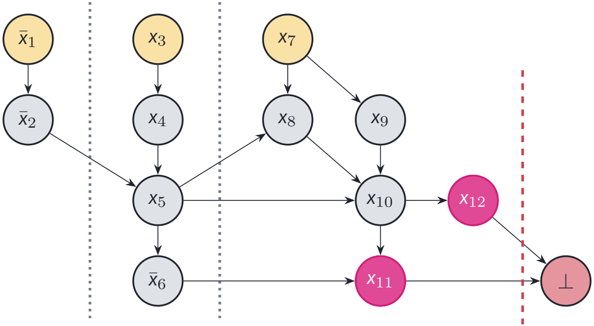


$x_1 \vee \bar{x}_3 \vee \bar{x}_7$ (all decisions: the “nogood”)



$x_6 \vee \bar{x}_{10}$ (first unique implication point, 1-FUIP)

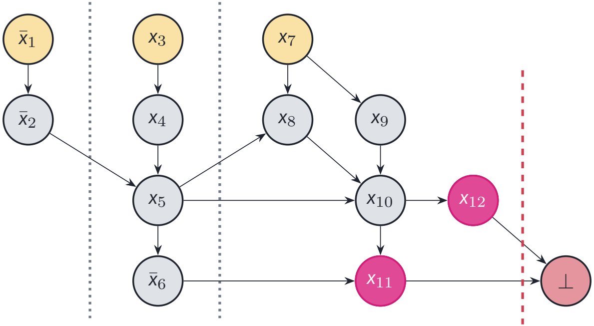
Conflict analysis: resolution view



$$C_9 : \bar{x}_{10} \vee x_{12}$$

$$C_{10} : \bar{x}_{11} \vee \bar{x}_{12}$$

Conflict analysis: resolution view



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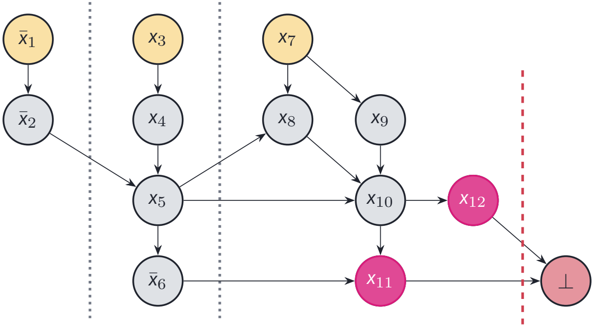


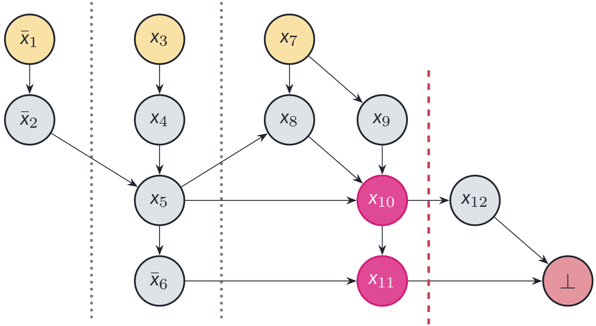
$C_9 : \bar{x}_{10} \vee x_{12}$

$C_{10} : \bar{x}_{11} \vee \bar{x}_{12}$

Resolution rule

$p \vee F_1$ and $\bar{p} \vee F_2 \quad \Rightarrow \quad F_1 \vee F_2$

Conflict analysis: resolution view



$$C_9 : \bar{x}_{10} \vee x_{12}$$

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$$\Downarrow$$

$$R_{9,10} : \bar{x}_{10} \vee \bar{x}_{11}$$

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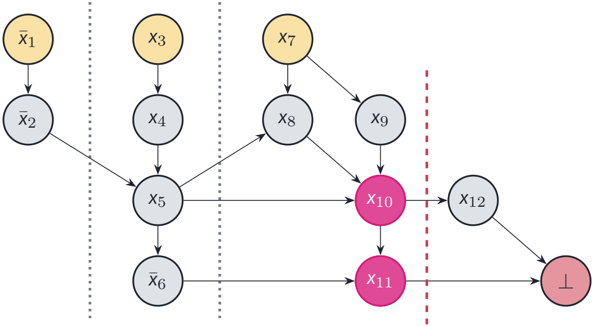

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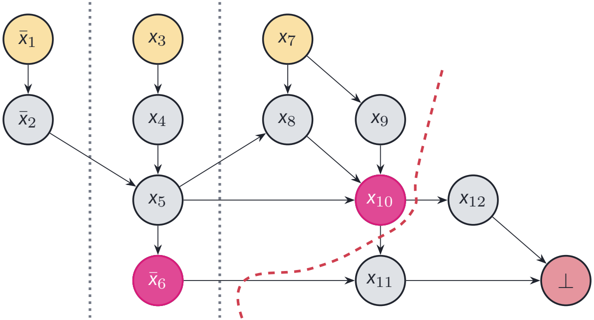

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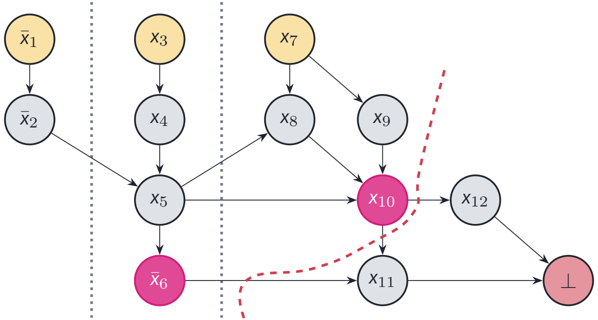

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Conflict analysis = resolve bound changes in reverse order until a stopping criterion (e.g., 1-FUIP).

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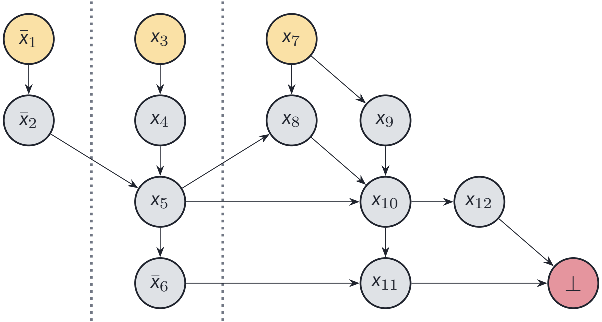
- A Chvátal–Gomory cut (or coefficient tightening) yields

$$x_2 + (1 - x_3) + x_4 \geq 1 \iff x_2 \vee \bar{x}_3 \vee x_4$$

Resolution $=$ aggregation $+$ cut strengthening. Keep this in mind for MIP!

Flavour 1: Graph-Based Conflict Analysis

First generalization to MIP



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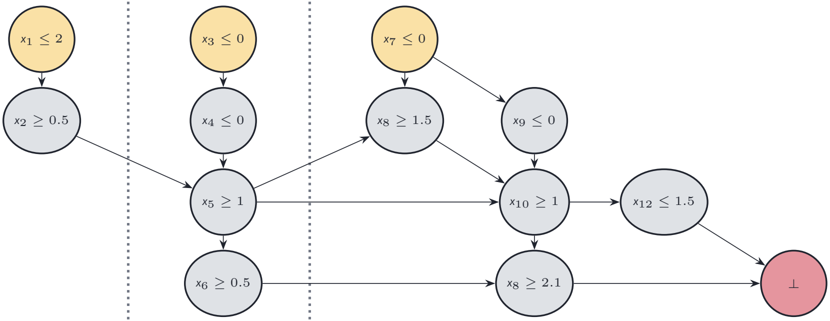

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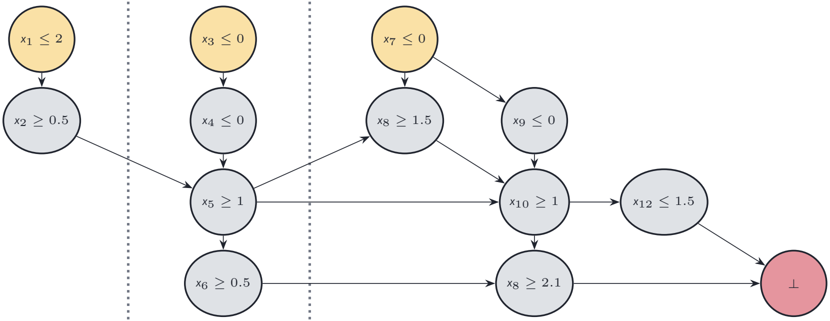

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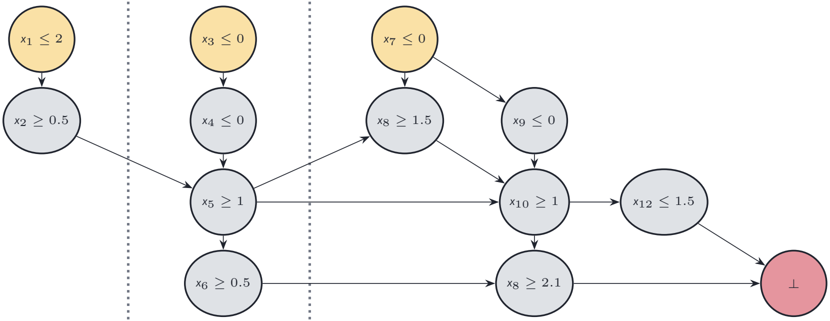

- Introduced independently by Achterberg (2007) and Sandholm and Shields (2006)
- Vertices are now bound changes $x_j \geq \beta$ / $x_j \leq \beta$ instead of assignments
- A cut in the graph yields a bound disjunction, e.g., $(x_1 \geq 3) \vee (x_3 \geq 1) \vee (x_7 \geq 1)$
  - usable directly if the solver supports disjunctive constraints
  - linearizable if all involved variables are binary ($\rightarrow$ clause inequality)

Flavour 2: Cut-Based Conflict Analysis

- Graph-based analysis in MIP learns clauses / bound disjunctions — it forgets the linear structure of the constraints that produced the conflict.

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- Pseudo-Boolean (PB) solvers (Chai and Kuehlmann, 2005) do conflict analysis directly on linear 0-1 constraints: much more expressive, e.g., one inequality

$$x_1 + x_2 + \cdots + x_k \leq 1$$

replaces $\binom{k}{2}$ clauses $\bar{x}_i \vee \bar{x}_j$.

Fundamental problem

Given:

- ◇ two rows C_{confl} and C_{reason},
- ◇ a local domain L (the current bound changes) and a global domain G,
- ◇ such that $\{C_{\text{confl}}, C_{\text{reason}}, L\}$ is infeasible.

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Conflict analysis $=$ solve the fundamental problem iteratively along the bound-change stack.

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- resolution:

$$\frac{C_{\text{reason}} \qquad C_{\text{confl}}}{P \vee Q}$$

  remains infeasible under L $\quad\rightsquigarrow\quad C_{\text{learn}} = P \vee Q$ ✓

Now try the same with linear constraints …

- Suppose somebody propagated $x_2 = 0$ and $x_3 = 1$, and we have $2x_1 + x_2 + x_3 \geq 2$ and $-2x_1 + x_2 \geq -1$.

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- So we aggregate to cancel x_1: $\quad 2x_2 + x_3 \geq 1$.
- But this is feasible under $x_2 = 0, x_3 = 1$ ... what is going on?

- Under $x_2 = 0,\ x_3 = 1$ the reason $2x_1 + x_2 + x_3 \geq 2$ becomes

$$2x_1 + 1 \geq 2 \iff x_1 \geq \frac{1}{2}$$

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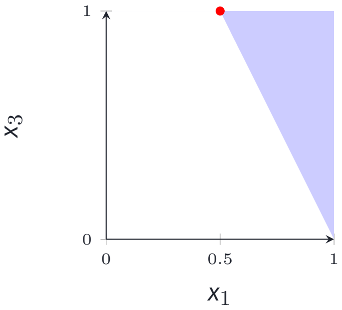

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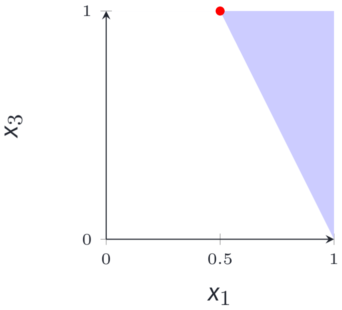
- non-tight propagation $\Rightarrow$ fractional vertex



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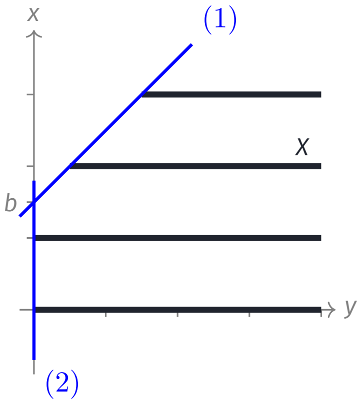
Fix: cut off that fractional vertex — reduce the reason so that it propagates tightly.

The cut we use for reduction (Marchand and Wolsey, 2001). Simplest case — the elementary mixed-integer set in two variables:

$$X := \{\ (x, y)\ \in\ \mathbb{Z} \times \mathbb{R} : \begin{array}{rcll} x & \leq & b + y & (1) \\ y & \geq & 0 & (2)\ \} \end{array}$$



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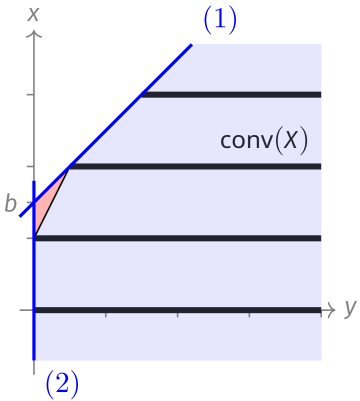

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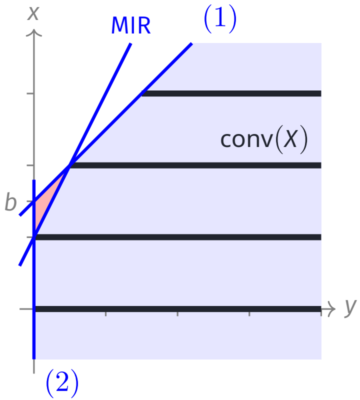
Rounding b down is not valid — but paying $\frac{1}{1-f(b)}$ per unit of y is:

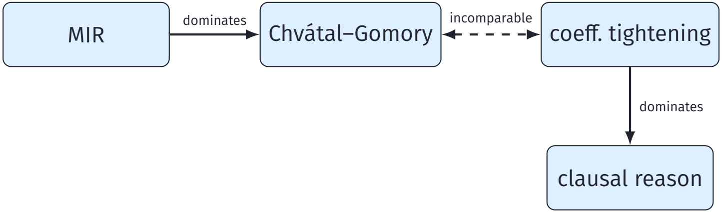
$$\underbrace{x\ \leq\ \lfloor b \rfloor + \frac{1}{1-f(b)}\, y}_{\textbf{MIR inequality}}$$

with $f(b) = b - \lfloor b \rfloor$ the fractional part of b

(1), (2) and the MIR inequality describe $\mathrm{conv}(X)$.



Several reductions restore tight propagation — which gives the tightest learned constraint?



Example: generalized resolution with MIR

$$
\begin{aligned}
C_{\text{reason}} : \quad & x_1 + x_2 + 2x_3 \geq 2 \\
C_{\text{confl}} : \quad & x_1 - 2x_3 + x_4 + x_5 \geq 1
\end{aligned}
$$

$\rho = \{\, x_1 \overset{\text{dec.}}{=} 0,\; x_3 \overset{C_{\text{reason}}}{=} 1 \,\} \Rightarrow$ conflict with C_{confl}

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Plain resolution on x_3 fails (as before). Complement x_2 and reduce:

$$\text{MIR}(C_{\text{reason}}) \quad \frac{x_1 + x_2 + 2x_3 \geq 2}{x_1 + x_3 \geq 1}$$

$$\text{resolve}\{x_3\} \quad \frac{x_1 + x_3 \geq 1 \qquad x_1 - 2x_3 + x_4 + x_5 \geq 1}{3x_1 + x_4 + x_5 \geq 3}$$

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Mixed-Integer:

- × With general integers there are no guarantees

Consider $x_1, x_2 \in \{0, 1\}$, $y_1, y_2 \in \mathbb{R}$, and

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C_1 &: \; x_2 + y_2 \geq 0 \\
C_2 &: \; -2x_1 - 4y_1 - 2y_2 \geq -3 \\
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C_4 &: \; -20x_1 + 5y_1 - 10y_2 \geq -16
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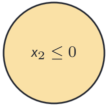

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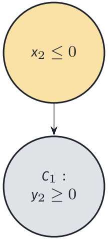

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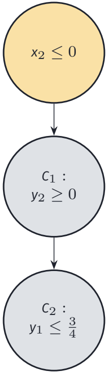

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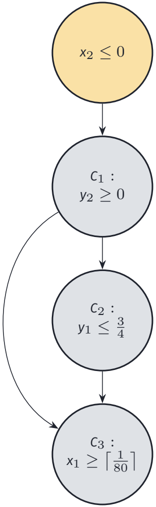

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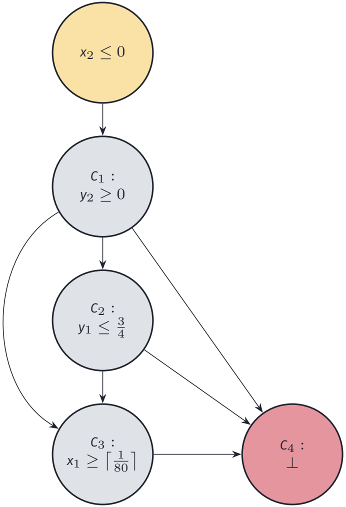

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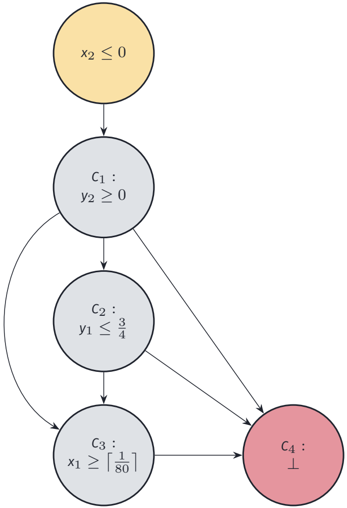

Consider $x_1, x_2 \in \{0, 1\}$, $y_1, y_2 \in \mathbb{R}$, and

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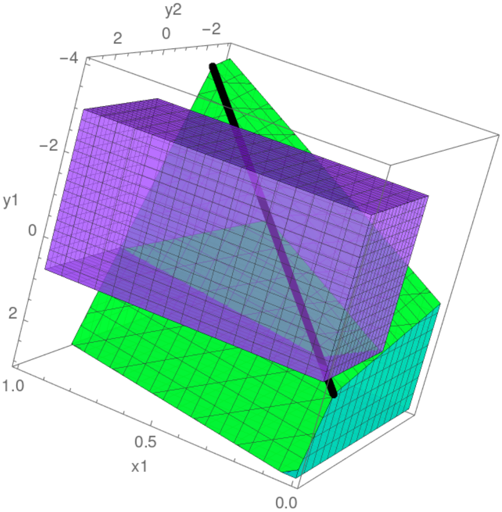
The infeasible local domain $x_2 = 0$, $y_1 \leq \frac{3}{4}$, $y_2 \geq 0$ under C_3, C_4 cannot be separated by any inequality valid for $\{C_3, C_4, G\}$!

Mixed-binary example geometry



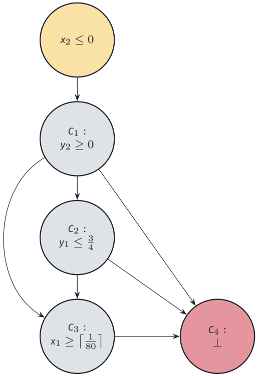
- We had

$$
\begin{aligned}
C_1 &: \; x_2 + y_2 \geq 0 \\
C_2 &: \; -2x_1 - 4y_1 - 2y_2 \geq -3 \\
C_3 &: \; 20x_1 + 5y_1 - y_2 \geq 4 \\
C_4 &: \; -20x_1 + 5y_1 - 10y_2 \geq -16
\end{aligned}
$$

- x_1 could not be resolved: the local bounds on y_1, y_2 got in the way.

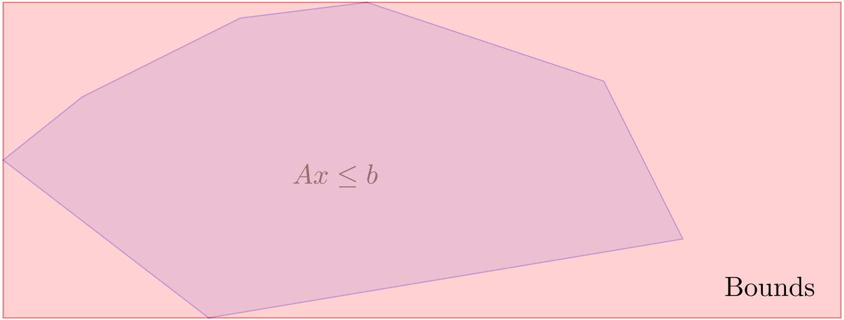
Try it yourself

Aggregate C_1, C_2, C_3 so that both continuous variables cancel, and check whether the resulting reason still propagates x_1.



Flavour 3: Dual Proof Analysis

So far the infeasibility came from propagation of a single constraint. What if the whole LP relaxation $\{Ax \geq b,\ \ell' \leq x \leq u'\}$ is infeasible under the local bounds?



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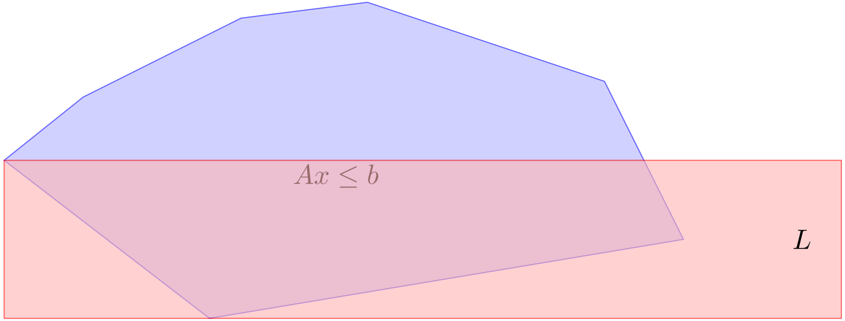

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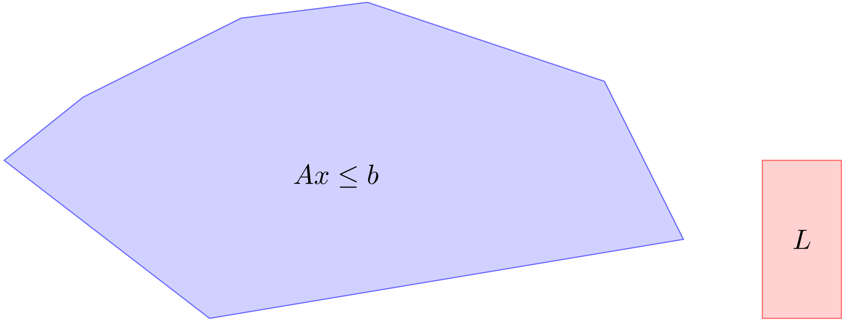

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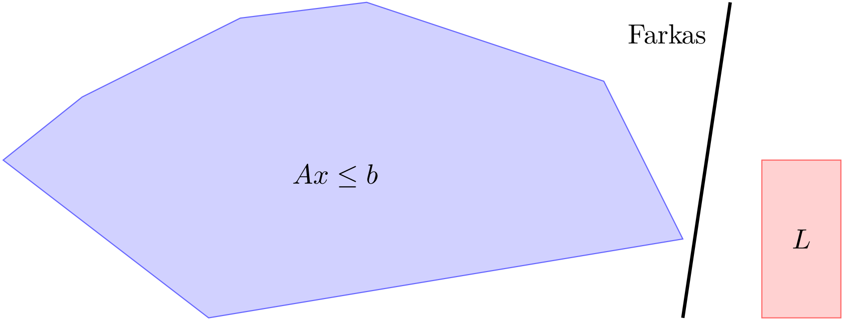

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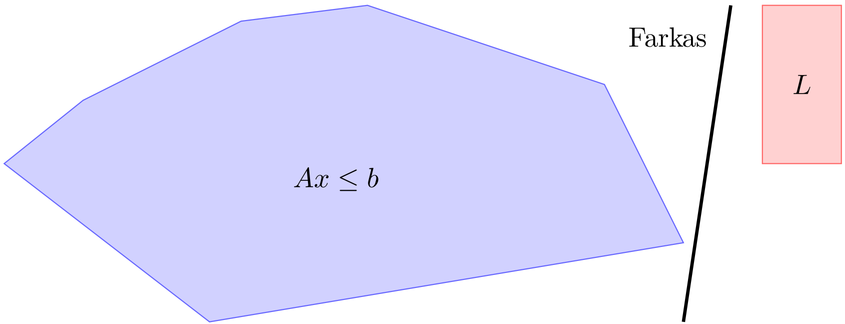

Infeasible node LP: $\min\{c^\mathsf{T}x \;:\; Ax \geq b,\ \ell' \leq x \leq u'\}$

- By Farkas' lemma there is a dual ray $y \geq 0$ aggregating the rows into

$$y^\mathsf{T}A\,x \;\geq\; y^\mathsf{T}b$$

which already contradicts the local bounds $[\ell', u']$ (max-activity $< y^\mathsf{T}b$).

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Dual proof (Farkas) analysis

Infeasible node LP: $\min\{c^{\mathsf{T}}x \ : \ Ax \geq b,\ \ell' \leq x \leq u'\}$

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Two ways to use it:

- find a smaller subset of local bound changes keeping the proof violated
  $\rightarrow$ run graph-based analysis from subset
- Witzig et al. (2019): keep the linear constraint itself as the learned constraint; strengthen it by coefficient tightening / MIR

- Conflict analysis learns globally valid constraints that separate infeasible local domains
- Graph-based: SAT machinery on bound changes; learns bound disjunctions.
- Cut-based: Generalized resolution on the original rows; learns a linear row.
- Dual proof: Farkas aggregation of an infeasible LP; learns a linear row.
- Worth it: switching conflict analysis off costs a modern solver $\sim$30 % performance.

Interested in a student job in AI and Optimization?

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Thank you!

Questions?

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